Chapter 10: Brownian Motion and Related Processes

- Recall the symmetric random walk, where at each time unit we (with equal probability) take either a one-unit step to the left or a one-unit step to the right.
- Suppose we let both the time interval widths Δt and the step size Δx get smaller and smaller.
- Then if $X(t)$ is our position at time t, and

$X_i =$

then $X(t) =$

where $[t/\Delta t]$ is the largest integer less than or equal to $\frac{t}{\Delta t}$.

- Here we assume that the X_i's are independent and

- Note $E(X_i) =$ and $E(X_i^2) =$

$\Rightarrow$ $\text{var}(X_i) =$

- Thus $E[X(t)] =$ and $\text{var}[X(t)] =$

- We let Δx and Δt get small in the following way:

- Take $\Delta x = \sigma\sqrt{\Delta t}$ and let $\Delta t \to 0$

Then $E[X(t)] =$ and $\text{var}[X(t)] \to$

- By the Central Limit Theorem, as $\Delta t \to 0$, $X(t)$ is ____________ with mean and variance

- We call the resulting process

____________________.

__Defn.__ A stochastic process $\{X(t), t \geq 0\}$ is called __Brownian motion__ (also known as the __Wiener process__) if

(i)

(ii)

(iii)

- This was originally conceived by Robert Brown as a model for the motion of a particle immersed in a liquid or gas.
- This motion was later explained by Albert Einstein and the process was formally defined by Norbert Wiener.
- If $\sigma = 1$, the process is ________ Brownian motion.
- We can always standardize any Brownian motion process $\{X(t)\}$ by taking $B(t) =$ so without loss of generality, we will usually assume that $\sigma = 1$.

Note: $X(t)$ is (with probability 1) a continuous function of t, but is not differentiable for any t.

Intuitive "Proof":

Distribution of $X(t)$

- If $\sigma = 1$, $X(t) \sim$ so that its pdf is

- Often of interest is the conditional distribution of $X(s)$, given that $X(t) = B$, where $s < t$.
- It can be shown (see pg. 609-610) that this conditional distribution is

__________ with

$E[X(s) \mid X(t) = B] =$

$\text{var}[X(s) \mid X(t) = B] =$

Example 1: Suppose the change in a stock price over time is modeled as standard Brownian motion. What is the conditional distribution of $X(20)$ given that $X(15) = 3.5$?

- What is the conditional distribution of $X(10)$ given that $X(15) = 3.5$?

10.2 Hitting Time Results

- Let T_a be the first time the Brownian motion process hits the value a.
- Suppose $a > 0$. Then for some time $t > 0$,

since if

Now,

- Consider the distribution of the maximum value the process hits in $[0, t]$. For $a > 0$,

Gambler's Ruin Revisited

- If a stock price follows Brownian motion, what is the probability that, starting at 0, it hits A before it hits $-B$? (goes "up by A" instead of going "down by B")
- Recall the gambler's ruin results for the symmetric ($p = \frac{1}{2}$) random walk.

- The "step sizes" (like dollars) are in Brownian motion:

<u>Example:</u> A stock is initially offered at \$10 per share and its price then follows Brownian motion. What is the probability that the price hits \$15 before it hits \$7?

10.3 Variations on Brownian Motion

Defn: $\{X(t)\}$ is Brownian motion with drift coefficient μ and variance σ^2 if

(i)

(ii)

(iii)

Note: If $\{B(t)\}$ is standard Brownian motion and $X(t) =$

then $\{X(t)\}$ is Brownian motion with drift.

Geometric Brownian Motion

- If $\{Y(t)\}$ is Brownian motion with drift μ and variance σ^2, then

- Geometric Brownian motion assumes the percentage changes of $X(t)$ over time are iid.