10.5 Maximum of Brownian Motion with Drift

- In Section 10.2 we derived the probability distribution of the maximum value within $[0,t]$ of a standard Brownian motion process.
- Now we consider Brownian motion with drift coefficient μ and variance parameter σ^2.
- Let $M(t) = \max_{0 \le y \le t} X(y)$ be the maximum value up to time t.
- We will derive the distribution of $M(t)$ by conditioning on $X(t)$.

Lemma: If $Y_1, \dots, Y_n \overset{iid}{\sim} N(\theta, \nu^2)$, then the conditional distribution of $Y_1, \dots, Y_n$ given $\sum_{i=1}^{n} Y_i = x$ does not depend on θ.

<u>Note</u>: This says that in a sample from a normal distribution, the sample sum is a ________________ for the population mean.

- This is proved in STAT 512, but we will not prove it here.

<u>Theorem</u>: Let $\{X(t)\}$ be Brownian motion with drift coefficient μ and variance parameter σ^2. Given that $X(t) = x$, the conditional distribution of $X(y)$, for $0 \le y \le t$, does not depend on μ.

<u>Proof</u>: For a fixed n, let $t_i =$

By the Lemma,

Conditional Distribution of $M(t)$

Theorem: For $y > x$,

$$P(M(t) \geq y \mid X(t) = x) = e^{-2y(y-x)/t\sigma^2}, \quad y \geq 0$$

Proof:

- Let some small $h > 0$ be such that $y > x + h$. Then

Thus

<u>Corollary</u>: If $\Phi(\cdot)$ is the standard normal cdf, then

$$P(M(t) \geq y) = e^{2y\mu/\sigma^2}\left[1 - \Phi\left(\frac{\mu t + y}{\sigma\sqrt{t}}\right)\right] + \left[1 - \Phi\left(\frac{y - \mu t}{\sigma\sqrt{t}}\right)\right]$$

Proof: Condition on $X(t)$ and apply the previous theorem.

$$P(M(t) \geq y) = \int_{-\infty}^{\infty} P(M(t) \geq y \mid X(t) = x) f_{X(t)}(x)\,dx.$$

Some calculus (see pg. 627-628 for details) yields the result.

Note: The last result in Section 10.2 is a special case of this result, with $\mu = 0$ and $\sigma^2 = 1$.

Example 1 again: A change in stock price follows standard Brownian motion. What is the probability that the stock price will increase by at least \$10 in the first $t = 25$ days?

Example 2: Suppose the change in price of a commodity follows Brownian motion with drift = 0.5 and $\sigma^2 = 4$. What is the probability that the price does not increase by more than \$12 in the first 64 days?

10.6 Stochastic Integrals and White Noise

- Let $\{X(t)\}$ be standard Brownian motion and let $f(\cdot)$ be a function with a continuous derivative over $[a,b]$.

Defn. The stochastic integral

where $a = t_0 < t_1 < \cdots < t_n = b$.

Note: By using the following identity (which is the integration by parts formula with sums replacing integrals and differences replacing differentials):

we may rewrite the definition of the stochastic integral:

$\int_a^b f(t)\,dX(t) =$

<u>Note</u>: The stochastic integral idea was introduced by Kiyoshi Itô in the 1940s. It is sometimes called the <u>Itô integral</u>.

- Assuming that expectation and limit can be interchanged, since $E[X(t)] =$ for any t,

$E\left[\int_a^b f(t)\,dX(t)\right] =$

And $\operatorname{var}\left\{\sum_{i=1}^{n} f(t_{i-1})\left[X(t_i) - X(t_{i-1})\right]\right\}$

$=$

Taking limits as $n \to \infty$, we have

<u>Note</u>: We may view the $\{dX(t), t \geq 0\}$ as an operator that maps functions $f(t)$ to the stochastic integral $\int_a^b f(t)\, dX(t)$.

- This is called a <u>white noise transformation</u> and $\{dX(t), t \geq 0\}$ is called <u>white noise</u>.

Example (Habitat size): Suppose the rate of change of a habitat size gets randomly smaller as the habitat gets larger, according to the formula:

$$S'(t) = -2S(t) + \sigma X'(t)$$

where $X'(t) = dX(t)$ is white noise. If the initial habitat size is 100 acres, find a formula for the habitat size at time t if $\sigma = 1$.