10.7 Gaussian Processes

Defn: A stochastic process $\{X(t), t \geq 0\}$ is a *Gaussian process* if $(X(t_1), \dots, X(t_n))$ has a multivariate normal distribution for all $t_1, \dots, t_n$.

Example: Let $\{X(t)\}$ be Brownian motion. Then $X(t_1), X(t_2), \dots, X(t_n)$ can each be expressed as a linear combination of

which are __________ ________ r.v.'s, hence $(X(t_1), \dots, X(t_n))$ is __________

________ and $\{X(t)\}$ is a __________

________.

Note: A multivariate normal distribution is completely determined by the marginal means $E(X_i)$ and the set of covariance elements $\mathrm{cov}(X_i, X_j)$ (including the variance components).

- For standard Brownian motion, $E[X(t)] =$ for all t, and for $s \le t$, $\mathrm{cov}[X(s), X(t)] =$

- Thus standard Brownian motion could be defined as a Gaussian process with $E[X(t)] =$ for $t \ge 0$ and $\mathrm{cov}[X(s), X(t)] =$ for $s \le t$.

The Brownian Bridge

- Let $\{X(t)\}$ be standard Brownian motion, and suppose we know that $X(1) = 0$.
- Then the conditional stochastic process

$$\{X(t),\ 0 \le t \le 1 \mid X(1) = 0\}$$

is called the <u>Brownian bridge</u>.

Picture:

- Since $X(t_1), \ldots, X(t_n)$ is conditionally multivariate normal, the Brownian bridge is a ______________________

with conditional mean

$$E[X(s) \mid X(1) = 0] =$$

and conditional covariance, for $s < t < 1$,

<u>Theorem</u>: If $\{X(t)\}$ is standard Brownian motion, then $\{Z(t)\}$ is a Brownian bridge process when $Z(t) = X(t) - t\,X(1)$.

Proof:

Integrated Brownian Motion

Defn. If $\{X(t)\}$ is Brownian motion, let $Z(t) = \int_0^t X(s)\,ds$, and then $\{Z(t), t \geq 0\}$ is called integrated Brownian motion.

- Since $\{X(t)\}$ is a Gaussian process, $\{Z(t)\}$ can be shown to be a Gaussian process by writing this integral as a limit of approximating sums, where the components of the sums are independent normal r.v.'s.

- The mean of $Z(t)$ is

$$E[Z(t)] =$$

and for $s \le t$,

$$\mathrm{cov}[Z(s), Z(t)] =$$

Example: Suppose the inflation rate varies as Brownian motion, and suppose the price of a certain commodity has a rate of change equal to the current inflation rate. Find the correlation between the commodity price at $t=6$ days and the price at $t=12$ days.