Chapter 2: Random Variables

<u>Defn.</u> A <u>random variable</u> (r.v.) is a function that maps the set of outcomes in the sample space to a set of real numbers.

<u>Example 1</u> (two dice): Let the r.v. X be the sum of two fair dice. Then X is the function:

Outcome	X

<u>Defn.</u> A <u>discrete r.v.</u> is one that takes on a finite or countably infinite number of values.

<u>Defn</u>: A <u>continuous r.v.</u> is one that takes on a continuum of values.

<u>Example 2</u>: If the r.v. X measures the lifetime of a part, then X is

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<u>Defn</u>: The cumulative distribution function (cdf) of a r.v. X is defined as:

<u>Note</u>: $F(b)$ is a nondecreasing function of b.

$\lim_{b \to -\infty} F(b) =$ and $\lim_{b \to \infty} F(b) =$

<u>Also</u>: $P(a < X \leq b) =$

<u>2.2 Discrete Random Variables</u>

<u>Defn</u>. The probability mass function (pmf) of a discrete r.v. X is

- Note $p(a)$ is positive for a finite or countable number of values a.
- If X can take on values $x_1, x_2, \ldots$ then $\sum_{i=1}^{\infty} p(x_i) =$
- The cdf of a discrete r.v. X is

$$F(a) = \sum_{\text{all } x_i \le a} p(x_i)$$

2.3 Continuous Random Variables

- The probability density function (pdf) of a continuous r.v. X is a nonnegative function $f(x)$ such that

for any subset B of the real line.
- Note $P(a \le X \le b) =$

and $\int_{-\infty}^{\infty} f(x)\,dx =$

- The cdf of a continuous r.v. X is

and

- Note that for a small $\varepsilon > 0$,

2.4 Expectation of a Random Variable

- If X is a discrete r.v. with pmf $p(x)$, then the expected value of X is

- If X is a continuous r.v. with pdf $f(x)$, then the expected value of X is

Expectation of a Function of a r.v.

- If $g(X)$ is any function of a r.v. X, then

Special Cases: $E[aX+b] =$

for any constants a and b.

- The variance of X is defined as

2.6 Moment Generating Functions

Defn. The moment generating function (mgf) of a r.v. X is denoted $\phi(t)$ and defined as:

Note: The m-th derivative of $\phi(t)$, evaluated at $t=0$, equals $E[X^m]$, which is called the m-th moment of X.

So

- The mgf of a r.v. uniquely determines its distribution.

Table 2.1

Discrete probability distribution	Probability mass function, $p(x)$	Moment generating function, $\phi(t)$	Mean	Variance
Binomial with parameters n, p, $0 \le p \le 1$	$\binom{n}{x} p^x (1-p)^{n-x}$, $x = 0, 1, \ldots, n$	$(pe^t + (1-p))^n$	np	$np(1-p)$
Poisson with parameter $\lambda > 0$	$e^{-\lambda} \frac{\lambda^x}{x!}$, $x = 0, 1, 2, \ldots$	$\exp\{\lambda(e^t - 1)\}$	λ	λ
Geometric with parameter $0 \le p \le 1$	$p(1-p)^{x-1}$, $x = 1, 2, \ldots$	$\frac{pe^t}{1-(1-p)e^t}$	$\frac{1}{p}$	$\frac{1-p}{p^2}$

Table 2.2

Continuous probability distribution	Probability density function, $f(x)$	Moment generating function, $\phi(t)$	Mean	Variance
Uniform over (a, b)	$f(x) = \begin{cases} \frac{1}{b-a}, & a < x < b \\ 0, & \text{otherwise} \end{cases}$	$\frac{e^{tb} - e^{ta}}{t(b-a)}$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
Exponential with parameter $\lambda > 0$	$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & x < 0 \end{cases}$	$\frac{\lambda}{\lambda - t}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
Gamma with parameters (n, λ), $\lambda > 0$	$f(x) = \begin{cases} \frac{\lambda e^{-\lambda x}(\lambda x)^{n-1}}{(n-1)!}, & x \ge 0 \\ 0, & x < 0 \end{cases}$	$\left(\frac{\lambda}{\lambda - t}\right)^n$	$\frac{n}{\lambda}$	$\frac{n}{\lambda^2}$
Normal with parameters (μ, σ^2)	$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \times \exp\{-(x-\mu)^2/2\sigma^2\}$, $-\infty < x < \infty$	$\exp\left\{\mu t + \frac{\sigma^2 t^2}{2}\right\}$	μ	σ^2

- These common distributions are all familiar from STAT 511.
- Note the parameterization of the exponential and gamma here is different than in STAT 511:
- Here, the rate parameter λ is the same as $\frac{1}{\beta}$ where β was the scale parameter in the gamma.
- The shape parameter of the gamma is here denoted n (we used α in STAT 511).

2.5 Jointly Distributed r.v.'s

- Sometimes we are interested in probabilities involving more than one r.v.

Defn: If X and Y are r.v.'s, the joint cdf of X and Y is