Chapter 4: Discrete Markov Chains

- Suppose $\{X_n, n=0,1,2,\dots\}$ is a stochastic process whose set of possible values is finite or countable.

Notation: If the process is currently in state i, then the probability that it next will be in state j is denoted P_{ij}.

Defn. $\{X_n\}$ has the Markovian property if the probability the chain is in state j at the next time depends only on the state it is in at the current time, i.e.,

- This means the conditional distribution of X_{n+1} is independent of any states before the current state.
- The transition probabilities P_{ij} satisfy the usual properties:

- The entire set of one-step transition probabilities can be represented in matrix form:

Example 1: (Simple Markov chain). Let us assume that if it rains today, then it will rain tomorrow with probability 0.5. If it does not rain today, then it will rain tomorrow with probability 0.3. Find the transition probability matrix.

Example 2: Assume now that if it has rained yesterday and today, P[rain tomorrow] = 0.7. If it rained today, but not yesterday, P[rain tomorrow] = 0.5. If it rained yesterday but not today, P[rain tomorrow] = 0.4. And if it rains neither yesterday nor today, P[rain tomorrow] = 0.2. Is the two-state process indicating whether it rains on day n a Markov chain?

- Suppose we convert the process to

- What is the transition probability matrix?

Example 3: We have two urns, each of which contains 2 balls. Of the 4 total balls, 2 are white and 2 are black. At each step, we take one ball at random from each urn and place each selected ball in the opposite urn.

Let X_n be the number of white balls in the first urn after the n-th step.

- What is the state space of $\{X_n\}$?

- What is the transition probability matrix?

4.2 Chapman-Kolmogorov Equations

- We denote by P_{ij} the probability of the process moving from state i to state j in one step.

- Let P_{ij}^n denote the probability of moving from i to j in <u>n steps</u>, i.e.,

- The <u>Chapman-Kolmogorov</u> equations give a method to find such n-step probabilities:

<u>Proof:</u>

- Let $P^{(n)}$ denote the matrix containing the n-step transition probabilities P_{ij}^n, for all $i, j \geq 0$.

- Since the above equation represents a

and in particular,

In general, the n-step transition probability matrix $P^{(n)} =$, which can be shown easily by mathematical induction.

<u>Recall Example 1</u>: This was a simple two-state Markov chain with

- If it is raining today, what is the probability that it will rain five days from now?