4.3 Classification of States

Defn. State j is accessible from state i if $P_{ij}^n > 0$ for some $n \geq 0$.

- Hence if state j is accessible from state i, then if we start in state i, it is possible that the process will eventually enter j.

Defn. If states i and j are accessible to each other, then states i and j communicate.
[Shorthand: $i \leftrightarrow j$]

Note: States i and j communicate if and only if:

Note: Communication satisfies the properties:

(i)

(ii)

(iii)

Classes of States

Defn. Two states are in the same class if they communicate.

- The state space can thus be separated into separate classes of states.

Defn. A Markov chain is irreducible if it has only one class (all states communicate with each other).

- Recall Example 3: Transition probability matrix was:

- Is this chain irreducible?

Example 4 again: Transition probability matrix was:

- Is this chain irreducible?

- Thus state 3 is

- The classes of this Markov chain are:

<u>Example (#4.15)</u>: Consider the Markov Chain with transition probability matrix

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- What are the classes?

Recurrent and Transient States

- Let f_i be the probability that if we start in state i, we will <u>ever</u> reenter state i.

- If $f_i = 1$, then state i is ________.
- If $f_i < 1$, then state i is ________.
- Note that if the chain is in a <u>recurrent</u> state, then it will <u>certainly</u> return to the state eventually.
- Once it returns, it will eventually return again, etc.
- So if state i is recurrent, the chain will revisit state i ________.
- If state i is <u>transient</u>, then each time we are in state i, the probability the chain will <u>never</u> again return to state i is:
- Starting in state i, the probability the chain will be in state i for exactly n time periods is:
- This probability distribution is ________ and has mean

- Let the indicator

$I_n =$

- Then the number of time periods in state i is:

which is ____________ with expected value

- We can write this expected value as

- So state i is ____________ if

and is ____________ if

<u>Note</u>: Any transient state will only be visited

- Hence in a finite-state Markov chain, at least one state <u>must</u> be

- Otherwise, at some point the chain would run out of states to visit!

Corollary (Recurrence as a Class Property): If state i is recurrent and $i \leftrightarrow j$, then state j is recurrent.

Proof:

Implication 1: If one state in a class is recurrent, then all states in that class are:

Implication 2: All states of a finite irreducible Markov chain are

Example 3: Transition Probability Matrix

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 1 & 0 \end{bmatrix}$$

- This chain was ______________________, so

Example 5: Suppose Alan and Bob start playing each with \$2. After each game, the loser pays \$1 to the winner. In each game, P[Alan wins] = 0.6 and P[Bob wins] = 0.4. They quit playing after one player has lost all his money. Let X_n be Alan's capital after n games.

- Note the state space of $\{X_n\}$ is:

$P =$

- The classes of this chain are:

- Example #4.18 (One-dimensional Random Walk): Consider $\{X_n\}$ with state space $i = \ldots, -2, -1, 0, 1, 2, \ldots$ and transition probabilities $P_{i,i+1} = p$ and $P_{i,i-1} = 1-p$.

- Is this chain irreducible?
- Are the states recurrent or transient?

Solving this for α, we get