4.4 Long-Run Proportions and Limiting Probabilities

- Consider two states i and j. Let $f_{i,j}$ denote the probability that the Markov chain, starting in i, will ever enter j. That is:

Theorem: If i is recurrent and $i \leftrightarrow j$, then $f_{i,j} = 1$.

Proof:

More Notation: If state j is recurrent, let m_j denote the expected number of transitions until the chain returns to state j (after starting in j).

- Thus if

$$N_j =$$

then

Defn. Recurrent state j is positive recurrent if $m_j < \infty$ and null recurrent if $m_j = \infty$.

- Suppose the Markov chain is irreducible and recurrent. Let π_j denote the long-run proportion of steps that the chain is in state j.

Theorem: If a Markov chain is irreducible and recurrent, then no matter the initial state,

$$\pi_j = \frac{1}{m_j}$$

Proof: Suppose the chain starts in state i.

Note: State j is positive recurrent if and only if

Theorem: (positive recurrence as a class property). If i is positive recurrent and $i \leftrightarrow j$, then j is positive recurrent.

Proof:

Note: It follows trivially (by contradiction) that null recurrence is also a class property.

Corollary: An irreducible finite Markov chain is positive recurrent.

Proof:

Theorem: For an irreducible Markov chain that is positive recurrent, the long-run proportions are the unique solution to the equations

and

- If these equations have no solution, then the chain is either transient or null recurrent and all $\pi_j =$

Example 1 again: What are the long-run proportions of rainy days (state 0) and non-rainy days (state 1)?

Example #4.22: Consider a model where successive generations of a family attain upper (0), middle (1), or lower (2) class status based only on their parents' status, with transition probability matrix:

So the long-run proportion of people in each social class can be found by:

<u>Note</u>: These long-run proportions π_j are called the <u>stationary</u> <u>probabilities</u> of the Markov chain.

Theorem: If the initial state is chosen according to these stationary probabilities, i.e., $P[X_0 = j] = \pi_j$ for $j \geq 0$, then at *any* time n,

$$P[X_n = j] = \pi_j \quad \text{for } j \geq 0.$$

Proof

Example (#4.25): Suppose the number of families who check into a hotel each day is Pois (λ). Also, the number of nights each family stays in the hotel is geometric (p), where $p \in (0,1)$ represents each family's check-out probability each day. Also, assume families act independently of each other.

Let X_n = the number of families checked in at the beginning of day n.

- What are the transition probabilities of the Markov chain $\{X_n\}$?

<u>Note</u>: If i families are currently checked in, the number of these families who will remain another day is a

________________ r.v., where