4.7 Branching Processes

- Branching processes are Markov chains that model changes in population size over time.
- Assume the individuals in the population produce offspring as follows:
- Each individual will in its lifetime produce j offspring with probabilities

  $P_j \quad (j = 0, 1, 2, \ldots)$.
- Each individual produces offspring independently of other individuals.
- The initial size of the population is X_0 (the size of the zero-th generation).
- Let X_1 be the total number of offspring of the entire zero-th generation.

- In general:

  X_n = size of the n-th generation
  = total offspring of the (n-1)-st generation

- What is the state space of the Markov chain $\{X_n\}$?

- Are the states recurrent or transient?

Corollary: If $P_0 > 0$, then eventually the population will die out or its size will converge to infinity.

Proof:

Mean and Variance of X_n

- Let $\mu =$ denote the expected number of offspring for a random individual.
- Let $\sigma^2 =$ denote the variance of the number of offspring of an individual.
- Assume for now that the initial population size $X_0 = 1$. Note

$X_{n-1} =$

- Let Z_i be the number of offspring of the i-th individual in the (n-1)-st generation.

Then

- By the law of iterated expectation:

By the law of iterated variance:

Extinction Probabilities

- Let π_0 denote the probability the population will eventually die out (assuming $X_0 = 1$):

$$\pi_0 =$$

<u>Theorem</u>: If $\mu < 1$, then $\pi_0 = 1$.

<u>Proof</u>:

<u>Note</u>: If $\mu = 1$, it can be shown that $\pi_0 = 1$.

What if $\mu > 1$? We condition on the number of offspring of the initial individual:

- If there are j individuals in the first generation, then the overall population dies out if and only if:

- Each independent branch has probability of dying out, so overall:

Thus

- When $\mu > 1$, then π_0 equals the smallest positive number satisfying this equation.

<u>Example 1</u>: Suppose women of a certain population will have j <u>female</u> offspring with the following probabilities, for $j = 0, 1, 2, 3, 4, 5$: $P_0 = 0.300$, $P_1 = 0.445$, $P_2 = 0.165$, $P_3 = 0.060$, $P_4 = 0.022$, $P_5 = 0.008$.

- Suppose there is initially one female in such a family. Find $E(X_n)$ and $\mathrm{var}(X_n)$.

- What is the probability of the family dying out?

- What is the probability of extinction if the family initially has 3 females?