Chapter 5: The Exponential Distribution and the Poisson Process

- The exponential distribution is a common model for lifetimes or waiting times.

- It has the memoryless property: An item whose lifetime is exponential has the same chance (no matter how old it is) of lasting a certain further amount of time as does a new item.

Defn. A continuous r.v. X has an exponential distribution with rate $\lambda > 0$ if its pdf is:

- The cdf of an exponential r.v. is:

- The mgf of an exponential r.v. is:

- From this, we see the mean of the exponential distribution is:

Memoryless Property

Defn. A r.v. X is memoryless if

- That is, in the context of lifetimes, the chance of an item lasting s additional time units is the same no matter the item's current age t.
- The item "does not remember" that it has already lasted t time units.

Theorem: An exponential r.v. X is memoryless.

Proof:

Example 1: You are waiting for service at a post office. Suppose the waiting time X is exponential with mean 5 minutes ($\lambda =$). What is the probability you will wait more than 10 minutes?

- If you have already been waiting 7 minutes, what is the probability that your total wait will be more than 15 minutes?

Example 1(a): Same post office, but now there are two clerks. Suppose each clerk has service time $X \sim \text{expon}(\lambda = \ )$. The two clerks are each serving a customer, and you are the only customer waiting. What is the probability that of the three customers in the store, you will be the last to be finished being served?

Example # 5.4 Suppose the monetary damage of a random car accident is exponential with mean \$1000. The insurance company will pay any amount exceeding the deductible (which is \$400). Find the expected value and the standard deviation of the payout.

Note: The exponential distribution is the only continuous distribution with the memoryless property.

Proof: The memoryless property implies

Since g is continuous, this implies

Note: The geometric distribution is the only discrete distribution that has the memoryless property.

More Properties of the Exponential

Theorem: If $X_1, \ldots, X_n$ are iid exponential r.v.'s, each with rate λ, then $X_1 + \cdots + X_n$ has a distribution.

- This is easily proved by using mgf's or by mathematical induction.

Example: Let X_1 and X_2 be independent exponential r.v.'s with rates λ_1 and λ_2. Find $P(X_1 < X_2)$.