5.3 The Poisson Process

Defn. A stochastic process $\{N(t), t \geq 0\}$ is called a counting process if $N(t)$ represents the total number of events that have occurred by time t.

Example 1: $N(t)$ = total number of customers entering a store by time t.

Example 2: $N(t)$ = total number of people in a population who have been born by time t.

Properties of a Counting Process

(1) $N(t) \geq 0$ for all t

(2) $N(t)$ is integer-valued.

(3) If $s < t$, then $N(s) \leq N(t)$.

(4) For $s < t$, then $N(t) - N(s)$

$=$

Note: "Number of customers in a store at time t" and "Number of people alive at time t" are not counting processes. Why not?

- A counting process has independent increments if the numbers of events occurring in disjoint time intervals are independent.
- For example, if

- Which example (1 or 2) is more likely to have independent increments?
- A counting process has stationary increments if the distribution of the number of events in an interval depends only on the length of the interval, i.e., if for any fixed $t>0$,

- Example 1 would have stationary increments only if:

Poisson Process

- An important counting process is the <u>Poisson</u> <u>Process</u>.

<u>Defn.</u> ("Little-oh" notation) A function $f(\cdot)$ is $o(h)$ if $\lim_{h \to 0} \frac{f(h)}{h} = 0$.

- Simply put, $f(\cdot)$ is $o(h)$ if $f(h)$ goes to 0 faster than h does.

<u>Examples</u>: $f(x) = x^2$ is

$f(x) = x$ is

- If $f(\cdot)$ is $o(h)$ and $g(\cdot)$ is $o(h)$, then $c_1 f(\cdot) + c_2 g(\cdot)$ is $o(h)$ for any constants c_1, c_2:

<u>Defn.</u> (Poisson Process): A counting process $\{N(t)\}$ is a <u>Poisson process</u> with rate $\lambda > 0$ if:

(i) $N(0) = 0$

(ii) $\{N(t)\}$ has independent increments

(iii) $P[N(t+h) - N(t) = 1] = \lambda h + o(h)$

(iv) $P[N(t+h) - N(t) \geq 2] = o(h)$

<u>Lemma</u> (Poisson approximation to binomial):

- Let $X \sim \text{binom}(n, p)$ and suppose $n \to \infty$ and $p \to 0$ such that $\lambda = np$ is constant. Then for $i = 0, 1, 2, \ldots$, Proof:

- Hence for n large and p small, a binomial probability can be approximated by a Poisson probability.

Theorem: If $\{N(t)\}$ is a Poisson process with rate $\lambda > 0$, then for all $s > 0$, $t > 0$,

$$N(s+t) - N(s) \sim \text{Poisson}(\lambda t).$$

- This implies the number of events in <u>any</u> interval of length t is Poisson(λt).

<u>Proof:</u> We first note: If we fix s and define $N_s(t)$ to be the count of events in the first t time units <u>past</u> s, then it is clear that

Interarrival and Waiting Time Distributions

- In a Poisson Process, let T_1 be the time of the first event.
- For $n=2,3,\ldots$, let T_n be the time elapsed between the $(n-1)$-st and n-th events.
- Then $\{T_n, n=1,2,\ldots\}$ is the sequence of interarrival times.
- Let's derive the distribution of T_1:

$$P[T_1 > t] =$$

- Hence T_1 is:
- And $P[T_2 > t] =$

So T_2 is

- This argument can be repeated for all $\{T_n\}$, showing:

<u>Theorem:</u> T_n $(n=1,2,\ldots)$ are iid exponential r.v.'s, each with rate λ.

<u>Note:</u> The assumption of <u>stationary</u> and <u>independent</u> increments implies that the process is <u>memoryless</u>, so having exponential interarrival times is unsurprising.

<u>Defn.</u> The <u>waiting time</u> until the n-th event (also called the <u>arrival time</u> of the n-th event) is defined as:

$$S_n = \sum_{i=1}^{n} T_i \qquad (\text{for } n=1,2,\ldots)$$

- Since the T_i's are iid exponential(λ), then $S_n \sim$ ________________ with pdf:

- Note also the cdf of S_n can be derived:

- Then differentiation of $F_{S_n}(t)$ yields the ________________ pdf.

<u>Example 1:</u> Suppose customers enter a store following a Poisson process with rate $\lambda = 25$ per hour.

- What is the probability of having exactly 3 customers in the first 15 minutes? How about <u>3 or fewer</u> in the first 15 minutes?

- What is the expected time until the 40th customer arrives?

- What is the probability that the next customer arrives within three minutes after this 40th customer?