

Chapter 7, continued: MANOVA

- The Multivariate Analysis of Variance (MANOVA) technique extends Hotelling T^2 test that compares two mean vectors to the setting in which there are $m \geq 2$ groups.
- We wish to compare the mean vectors across all m groups.
- For example, recall that in Fisher's iris data set, four measurements were made on irises from three different species.
- We could test whether the 4×1 mean vectors were equal for the three species (setosa, versicolor, virginica).
- Another example: Comparing the mean vectors (for several skull measurements) across the five epochs from which the skulls were found.
- MANOVA extends the ordinary ANOVA to the case in which $q \geq 2$ variables are measured on each observation.

The MANOVA Model

- If X_{ijk} represents the j -th observation on variable k in the i -th group ($k = 1, \dots, q$, $j = 1, \dots, n_i$, $i = 1, \dots, m$), then the ANOVA model decomposes X_{ijk} as:

$$X_{ijk} = \mu_k + \alpha_{ik} + \epsilon_{ijk}$$

- Here, μ_k is a general effect associated with the k -th variable, α_{ik} is the effect of group i on variable k , and ϵ_{ijk} is a random error term.
- In vector form, this is

$$\mathbf{X}_{ij} = \boldsymbol{\mu} + \boldsymbol{\alpha}_i + \boldsymbol{\epsilon}_{ij}, j = 1, \dots, n_i, i = 1, \dots, m$$

where these are each $q \times 1$ vectors.

- Similarly to the Hotelling test, we assume the vector $\boldsymbol{\epsilon}_{ij} = (\epsilon_{ij1}, \dots, \epsilon_{ijq})'$ is multivariate normal with mean vector $\mathbf{0}$ and common covariance matrix $\boldsymbol{\Sigma}$ across all m populations.
- In practice, the normality assumption can be checked via a chi-squared plot of the residual vectors $\mathbf{x}_{ij} - \bar{\mathbf{x}}_i$.

The MANOVA Null Hypothesis

- The mean vector for group i can be written as:

$$\boldsymbol{\mu}_i = (\mu_1 + \alpha_{i1}, \dots, \mu_q + \alpha_{iq})'$$

- Therefore, the MANOVA null hypothesis of equal mean vectors across the groups, i.e.,

$$H_0 : \boldsymbol{\mu}_1 = \boldsymbol{\mu}_2 = \dots = \boldsymbol{\mu}_m$$

can also be written as

$$H_0 : \alpha_{ik} = 0, i = 1, \dots, m, k = 1, \dots, q.$$

- The alternative hypothesis is that at least two of the groups' mean vectors differ, i.e., $\boldsymbol{\mu}_i \neq \boldsymbol{\mu}_{i'}$ for some $i \neq i'$.

Between-Groups and Within-Groups Matrices

- The MANOVA test statistic is based on two matrices, \mathbf{H} and \mathbf{E} :

$$\mathbf{H} = \sum_{i=1}^m n_i (\bar{\mathbf{x}}_i - \bar{\mathbf{x}})(\bar{\mathbf{x}}_i - \bar{\mathbf{x}})', \quad \mathbf{E} = \sum_{i=1}^m \sum_{j=1}^{n_i} (\mathbf{x}_{ij} - \bar{\mathbf{x}}_i)(\mathbf{x}_{ij} - \bar{\mathbf{x}}_i)'$$

- The matrix \mathbf{H} contains the between-groups sum of squares for each variable along its diagonal, and the between-groups sum of cross-products for the variables in its off-diagonal elements.
- The matrix \mathbf{E} contains the within-groups sum of squares for each variable along its diagonal, and the within-groups sum of cross-products for the variables in its off-diagonal elements.
- \mathbf{H} is the generalization of the treatment sum of squares (SSTR) in ordinary ANOVA, and \mathbf{E} is the generalization of the error sum of squares (SSE) in ordinary ANOVA.
- The matrix $\mathbf{H} + \mathbf{E}$ contains the total (corrected) sum of squares and cross products for the entire data set.

Test Statistic

- Unlike in ordinary ANOVA, where $F = MSTR/MSE$ gives the most powerful test of the ANOVA hypothesis, *no one test statistic* is uniformly most powerful in testing the MANOVA null hypothesis.

- The most common test statistic is Wilks' Lambda, a ratio of determinants:

$$\Lambda = \frac{|\mathbf{E}|}{|\mathbf{H} + \mathbf{E}|}$$

- Wilks' Lambda is related to the likelihood ratio criterion for this situation.
- A statistic that is a decreasing function of Λ has an F-distribution under H_0 .
- It corresponds to the F -statistic in the univariate situation and can be used via an F-test (or, for large samples, an approximate χ^2 test).

Other Test Statistics

- Other proposed test statistics include Roy's greatest root (the largest eigenvalue of $\mathbf{E}^{-1}\mathbf{H}$), the Lawley-Hotelling trace $tr(\mathbf{E}^{-1}\mathbf{H})$, or the Pillai trace $tr(\mathbf{H}(\mathbf{H} + \mathbf{E})^{-1})$.
- These also have approximate F-distributions under the null hypothesis.
- With large samples, these four test statistics are nearly equal and will give essentially equivalent conclusions.
- With two groups, these four test statistics are equal and are equivalent to the F-value in Hotelling's T^2 test.
- For moderate sample sizes, the Wilks, Lawley-Hotelling, and Pillai methods have similar power.
- Roy's method appears to be best only when there is a difference in just one component of the mean vector, and only one of the groups is different from the others.
- Pillai's trace may be more robust to non-normality, but transforming the data is suggested if the model residuals show departures from normality.

Multiple Comparisons in MANOVA

- If the F-test rejects the null that all m groups have equal mean vectors, we can use multiple comparisons to determine which specific pairs of groups have differing mean vectors (and which components of those mean vectors differ).
- This is most simply done via Bonferroni confidence intervals for all the differences $\alpha_{ik} - \alpha_{i'k}$ for all $i \neq i'$, where $i, i' \in \{1, \dots, m\}$.
- With family confidence level $1 - \alpha$, the set of CIs defined by:

$$(\bar{x}_{ik} - \bar{x}_{i'k}) \pm t_{n-m, \frac{\alpha}{qm(m-1)}} \sqrt{\frac{E_{kk}}{n-m} \left(\frac{1}{n_i} + \frac{1}{n_{i'}} \right)}$$

contain the $\alpha_{ik} - \alpha_{i'k}$.

- Any such interval that does not contain zero would indicate a significant difference in that component of the mean vector between that pair of groups.

Two-Way MANOVA

- We could have q response variables measured at various levels (categories) of two factors.
- Note that here, the factors are simply grouping variables, not the type of factors that we encountered in factor analysis!
- The one-way MANOVA model can be extended to a two-way model such as:

$$\mathbf{X}_{hij} = \boldsymbol{\mu} + \boldsymbol{\alpha}_h + \boldsymbol{\beta}_i + \boldsymbol{\gamma}_{hi} + \boldsymbol{\epsilon}_{hij},$$

where these are each $q \times 1$ vectors.

- Here, h indicates the level for the first factor, i indicates the level for the second factor, and j indicates the observation within the factor level combination.
- The vector $\boldsymbol{\gamma}_{hi}$ contains the effects for interaction between the two factors.

Tests in the Two-Way MANOVA

- When the random error vector is multivariate normal, tests about the interaction effects and factor effects can be carried out via F-tests.
- If the interaction effects are significant, then the effect of one factor is dependent on the level of the other factor.
- In this case, the F-tests for the factor effects are not applicable.
- Separate interaction plots for each variable can be done to investigate the nature of the interaction.
- If the interaction effects are not significant, then we can test whether the mean vectors differ across the levels of factor 1, and whether the mean vectors differ across the levels of factor 2.