A More Complicated Gibbs Example (Changepoint)

Then the posterior is $\pi(\lambda, \phi, k|\mathbf{x})$

$$\propto L(\lambda, \phi, k|\mathbf{x})p(\lambda)p(\phi)p(k)$$

$$= \left[\prod_{i=1}^{k} \frac{e^{-\lambda}\lambda^{x_i}}{x_i!}\right]\left[\prod_{i=k+1}^{n} \frac{e^{-\phi}\phi^{x_i}}{x_i!}\right]\left[\frac{\beta^\alpha}{\Gamma(\alpha)}\lambda^{\alpha-1}e^{-\beta\lambda}\right]\left[\frac{\delta^\gamma}{\Gamma(\gamma)}\phi^{\gamma-1}e^{-\delta\phi}\right]\left[\frac{1}{n}\right]$$

$$\propto e^{-k\lambda}\lambda^{\sum\limits_{i=1}^{k} x_i}e^{-(n-k)\phi}\phi^{\sum\limits_{k+1}^{n} x_i}\lambda^{\alpha-1}e^{-\beta\lambda}\phi^{\gamma-1}e^{-\delta\phi}$$

$$= \lambda^{\alpha+\sum\limits_{i=1}^{k} x_i - 1}e^{-(\beta+k)\lambda}\phi^{\gamma+\sum\limits_{k+1}^{n} x_i - 1}e^{-(\delta+n-k)\phi}$$

So full conditionals are:

$$\lambda|\phi, k \sim \text{gamma}(\alpha + \sum_{i=1}^{k} x_i, \beta + k)$$

$$\phi|\lambda, k \sim \text{gamma}(\gamma + \sum_{i=k+1}^{n} x_i, \delta + n - k)$$

A More Complicated Gibbs Example (Changepoint)

To get the full conditional for k, note the joint density of the data is:

$$
\begin{aligned}
p(\mathbf{x}|k,\lambda,\phi) &= \left[\prod_{i=1}^{k} \frac{e^{-\lambda}\lambda^{x_i}}{x_i!}\right]\left[\prod_{i=k+1}^{n} \frac{e^{-\phi}\phi^{x_i}}{x_i!}\right] \\
&= \left[\prod_{i=1}^{n} \frac{1}{x_i!}\right] e^{k(\phi-\lambda)} e^{-n\phi} \lambda^{\sum\limits_{i=1}^{k} x_i} \left[\prod_{i=k+1}^{n} \phi^{x_i}\right]\left[\frac{\prod\limits_{i=1}^{k} \phi^{x_i}}{\phi^{\sum_{i=1}^{k} x_i}}\right] \\
&= \left[\prod_{i=1}^{n} \frac{e^{-\phi}\phi^{x_i}}{x_i!}\right]\left[e^{k(\phi-\lambda)}\left(\frac{\lambda}{\phi}\right)^{\sum\limits_{i=1}^{k} x_i}\right] \\
&= f(\mathbf{x},\phi)g(\mathbf{x}|k)
\end{aligned}
$$

A More Complicated Gibbs Example (Changepoint)

By Bayes' Law, for any particular value k^* of k,

$$p(k^*|\mathbf{x}) = \frac{f(\mathbf{x}, \phi)g(\mathbf{x}|k^*)p(k^*)}{\sum_{k=1}^{n} f(\mathbf{x}, \phi)g(\mathbf{x}|k)p(k)}$$

Since $p(k) = 1/n$ (constant), we have

$$p(k^*|\mathbf{x}) = p(k^*|\mathbf{x}, \lambda, \phi) \propto \frac{g(\mathbf{x}|k^*)}{\sum_{k=1}^{n} g(\mathbf{x}|k)}$$

(full conditional for k)

- This ratio defines a probability vector for k that we use at each iteration to sample a value of k from $\{1, 2, \dots, n\}$.
- see R example (Coal mining data)

Another Gibbs Example (Normal Mixture)

Example 4 (Monkey Eye Data): $X_1, \ldots, X_{48}$ are a random sample of peak sensitivity wavelength measurements from a monkey's eyes (Bowmaker et al., 1985)

- The data are assumed to come from a mixture of two normal distributions, i.e.,

$$X_i \overset{\text{indep}}{\sim} N(\lambda_{T_i}, \tau) \text{ and } T_i \sim \text{Bernoulli}(p)$$

  where T_i ($= 1$ or 2) indicates the true group the i th observation came from.

- $\lambda_1 =$ mean of group 1, $\lambda_2 =$ mean of group 2, $\tau =$ common **precision** parameter (reciprocal of variance)
- For computational reasons, we let $\lambda_1 < \lambda_2$ and define the "mean shift" $\theta = \lambda_2 - \lambda_1$, $\theta > 0$.

Another Gibbs Example (Normal Mixture)

- We use the following independent noninformative priors on λ_1, θ, τ, and p:

$$
\begin{aligned}
p &\sim \text{beta}(1,1) \\
\theta &\sim N(0, \tau = 10^{-6}) I_{[\theta > 0]} \quad (\Rightarrow \sigma^2 = 10^6) \\
\lambda_1 &\sim N(0, \tau = 10^{-6}) \\
\tau &\sim \text{gamma}(0.001, 0.001)
\end{aligned}
$$

- Do example in WinBUGS with 1000-draw burn-in and then 10000 further draws.
- See convergence diagnostics in WinBUGS.