Homework 1 – STAT 704

1. Let $Y_{11}, \ldots, Y_{1n_1}$ be a sample from a population with mean μ_1 and variance σ_1^2. Let $Y_{21}, \ldots, Y_{2n_2}$ be a sample from another population with mean μ_2 and variance σ_2^2. Define

$$\bar{Y}_1 = \sum_{j=1}^{n_1} \frac{Y_{1j}}{n_1}, \quad \bar{Y}_2 = \sum_{j=1}^{n_2} \frac{Y_{2j}}{n_2}.$$

   (a) Find $E(\bar{Y}_1 - \bar{Y}_2)$.

   (b) If $\bar{Y}_1$ and $\bar{Y}_2$ are independent, find $var(\bar{Y}_1 - \bar{Y}_2)$.

   (c) If the two populations are normal (and $\bar{Y}_1$ and $\bar{Y}_2$ are independent), then does $\bar{Y}_1 - \bar{Y}_2$ have a normal distribution? Explain why or why not.

2. Suppose $Y_1, Y_2, \ldots, Y_n$ are independent random variables with mean μ and variance σ^2.

   (a) Show that

$$(n-1)S^2 = \sum_{i=1}^{n} Y_i^2 - n\bar{Y}^2.$$

   (b) Show that $E(S^2) = \sigma^2$. (Hint: Use the fact that $var(Y) = E(Y^2) - [E(Y)]^2$.)

3. Let Y_1, Y_2, Y_3 be independent random variables with means μ_1, μ_2, μ_3 and a common variance σ^2. Define

$$\bar{Y} = \frac{1}{3}\sum_{i=1}^{3} Y_i.$$

   (a) Find the covariance between $Y_1 - \bar{Y}$ and $\bar{Y}$.

   (b) Find the expected value of $(Y_1 + 2Y_2 - Y_3)^2$.

4. Let Y_1 and Y_2 be random variables with expected values μ_1 and μ_2 and variances σ_1^2 and σ_2^2.

   (a) Show that $cov(Y_1 + Y_2, Y_1 - Y_2) = \sigma_1^2 - \sigma_2^2$.

   (b) If $W = Y_1 + Y_2$ and $V = Y_1 - Y_2$, then under what condition(s) can we be assured that W and V are independent random variables?

5. Suppose $Y_1, \ldots, Y_5$ are independent random variables, each having a normal distribution with mean 0 and variance 1. Let $\bar{Y} = (1/5)\sum_{i=1}^{5} Y_i$ and let Y_6 be another independent observation, also having a N(0, 1) distribution. Let $W = \sum_{i=1}^{5} Y_i^2$ and let $U = \sum_{i=1}^{5}(Y_i - \bar{Y})^2$. Give the distribution of each of the following quantities, and carefully explain/justify your answer in each case.

   (a) W (b) U (c) $\sqrt{5}Y_6/\sqrt{W}$ (d) $2Y_6/\sqrt{U}$ (e) $20\bar{Y}^2/U$ (f) $4Y_6^2/U$ (g) $U/4Y_6^2$ (h) $\bar{Y} + Y_6$