STAT 511/MATH 511 – Test 1 – Summer 2021 Key

1. Consider a group of five applicants (three men and two women) for two identical jobs. The employer will select two of the five applicants to get the jobs. Denote the three men and two women by $\{M_1, M_2, M_3, W_1, W_2\}$. Let A be the subset of possibilities containing two men; let B be the subset of possibilities containing at least one woman.

   (a) List the elements of the following subsets: A, B, $A \cap B$, $A \cap \overline{B}$.

   $A = \{(M_1, M_2)\ (M_1, M_3)\ (M_2, M_3)\}$

   $B = \{(M_1, W_1)\ (M_2, W_1)\ (M_3, W_1)$
   $(M_1, W_2)\ (M_2, W_2)\ (M_3, W_2)\ (W_1, W_2)\}$

   $A \cap B = \emptyset$, $A \cap \overline{B} = \{(M_1, M_2)(M_1, M_3)(M_2, M_3)\}$

   (b) Regardless of gender, what is the total number of ways to select two people from the five candidates?

   $\binom{5}{2} = \frac{5!}{3!\,2!} = \frac{5 \cdot 4}{2} = 10$

   (c) If the two people selected to fill the jobs are selected completely at random, what is the probability that no women will be selected for the two jobs?

   $\frac{3}{10}$

2. Suppose that 8% of hydraulic landing assemblies have defects in shafts only; 5% of assemblies have defects in bushings only, and 2% have defects in both shafts and bushings. Suppose we randomly select one assembly. Show your work in finding the following:

   see next page also

   A = Shaft B = Bushing $P(A) = 0.08$ $P(B) = 0.05$ $P(A \cap B) = 0.02$

   (a) Find the probability it has either a shaft defect or a bushing defect (or both).

   $P(A \cup B) = 0.08 + 0.05 - 0.02$
   $= 0.11$

   (b) Find the probability it has exactly one of the two types of defects.

   $P(A \cup B) - P(A \cap B) = 0.11 - 0.02 = 0.09$

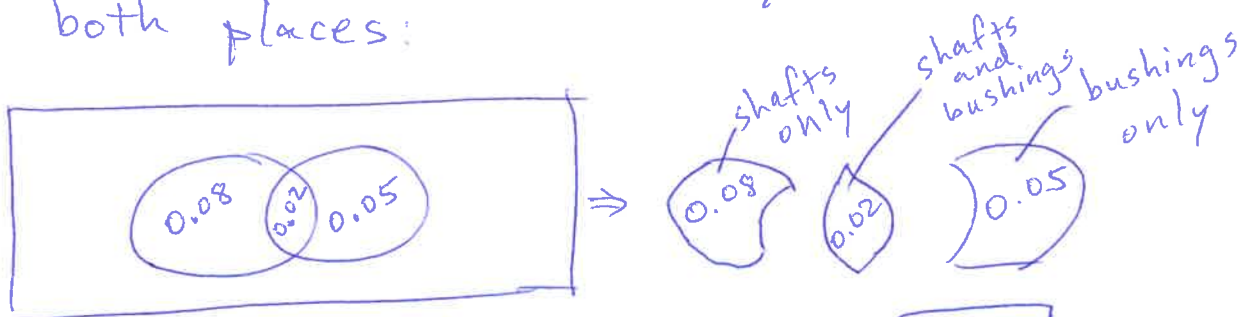
   

   (c) Find the probability it has neither type of defect.

   $P(\overline{A} \cap \overline{B}) = P(\overline{A \cup B}) = 1 - 0.11 = 0.89$

For Problem 2 on the first page, the provided solution would be correct if the word "only" were deleted from the two places it appears in the problem. (My intention was to not use the word "only" there, but I mistakenly did, so I regraded people's test with that in mind.) So here is the solution when the word "only" is included in both places:

(a)



$$P(A \cup B) = 0.08 + 0.05 + 0.02 = \boxed{0.15}$$

(b) $P(A \cap \bar{B}) + P(\bar{A} \cap B) = 0.08 + 0.05 = \boxed{0.13}$

(c) $P(\bar{A} \cap \bar{B}) = P(\overline{A \cup B}) = 1 - 0.15 = \boxed{0.85}$

3. Suppose a store sells two styles of wireless earbuds. Based on past experience, customers tend to prefer them equally. Suppose four customers in succession come into the store to buy earbuds. Consider the preferences of these four customers.

(a) Consider the sample space (i.e., all the possible preference arrangements of the four customers). How many sample points are in the sample space? What are the probabilities of the sample points?

16 sample points $\longrightarrow \frac{1}{16}$

(b) If A is the event that all four customers prefer the same style, find $P(A)$.

$$\frac{2}{16} = \frac{1}{8}$$

(c) If among the general population, the first style of earbuds is twice as popular as the second style of earbuds, in that case what is the probability that all four customers prefer the more popular style? Show how you got your answer.

$$\left(\frac{2}{3}\right)\left(\frac{2}{3}\right)\left(\frac{2}{3}\right)\left(\frac{2}{3}\right) = \frac{16}{81} = 0.198$$

4. Suppose that for a fixed dinner cost, a diner can select from four appetizers, three salads, four entrees, and five desserts. How many possible dinners are available if a dinner consists of one appetizer, one salad, one entree, and one dessert? Show how you got your answer.

$$4 \times 3 \times 4 \times 5 = 240$$

5. A restaurant needs to hire a cook, a server, and a cashier. Seven applicants have applied, all of whom want to be considered for any of the three job openings. How many different ways does the restaurant have to fill the openings? Show how you got your answer.

$$P_3^7 = \frac{7!}{4!} = 7 \cdot 6 \cdot 5 = 210$$

6. A student prepares for an exam by studying ten problems. He can solve five of the ten problems. For the exam, the teacher selects four of the ten problems to be on the exam.

(a) How many different sets of problems (ignoring order of the exam problems) could the teacher select? Show how you got your answer.

$$\binom{10}{4} = \frac{10!}{4!\,6!} = \frac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2 \cdot 1} = 210$$

(b) Assuming the teacher selects the exam problems at random from the ten total, then what is the probability that the student can solve all four problems on the exam? Show how you got your answer.

Label the problems the student can solve as 1, 2, 3, 4, 5.

Selections for which student can solve all problems:

$\{(1,2,3,4)\ (1,2,3,5)\ (1,2,4,5)\ (1,3,4,5)\ (2,3,4,5)\}$

$$\Rightarrow P[\text{solve all } 4] = \frac{5}{210} = \frac{1}{42} = \boxed{.0238}$$

7. Consider two events A and B such that $P(A) = 0.5$, $P(B) = 0.2$, and $P(A \cap B) = 0.1$.

(a) Find $P(B|A)$, $P(A|A \cap B)$, and $P(A \cap B|A \cup B)$. Show how you got your answers.

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{0.1}{0.5} = 0.2$$

$$P(A|A \cap B) = \frac{P(A \cap A \cap B)}{P(A \cap B)} = \frac{P(A \cap B)}{P(A \cap B)} = 1$$

$$P(A \cap B|A \cup B) = \frac{P[(A \cap B) \cap (A \cup B)]}{P(A \cup B)} = \frac{P(A \cap B)}{P(A \cup B)}$$

$$P(A \cup B) = 0.5 + 0.2 - 0.1 = 0.6$$

$$= \frac{0.1}{0.6} = \frac{1}{6} = \boxed{.1667}$$

(b) Are A and B independent events? Prove your answer.

Yes. $P(B|A) = 0.2 = P(B)$.

8. At a certain country club, 40% of the members play golf, 55% of the members play tennis, and 80% of the members play at least one of those two sports.

(a) If a member is chosen at random, what is the probability that member plays both golf and tennis? Show how you got your answer.

$$P(G \cup T) = P(G) + P(T) - P(G \cap T)$$

$$0.80 = .40 + .55 - P(G \cap T)$$

$$\Rightarrow P(G \cap T) = .40 + .55 - .80 = \boxed{0.15}$$

(b) For a randomly chosen member who is known to play golf, what is the probability that such a member plays tennis? Show how you got your answer.

$$P(T|G) = \frac{P(G \cap T)}{P(G)} = \frac{0.15}{0.40} = \boxed{0.375}$$

9. Suppose a student coming to statistics class walks to class 50% of the time, takes the bus 20% of the time, and rides her bike 30% of the time. She is late to class 15% of the time when she walks, is late 10% of the time when she rides the bus, and is late 5% of the time when she rides her bike. One Monday she is late to class. What is the probability that she walked to class that Monday? Show your work.

$$P(B_1|A) = \frac{P(A|B_1)P(B_1)}{P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + P(A|B_3)P(B_3)}$$

$$= \frac{(.15)(.50)}{(.15)(.50) + (.10)(.20) + (.05)(.30)}$$

$$= \frac{.075}{.075 + .02 + .015} = \frac{.075}{.11}$$

$$= \boxed{.6818}$$

10. In a typical season, the Gamecocks play four SEC football games at home (in Columbia). Suppose they have probability 0.25 of winning one home SEC game, 0.30 of winning two home SEC games, 0.20 of winning three home SEC games, and 0.05 of winning four home SEC games.

(a) If Y= the number of home SEC games they will win, give the complete probability distribution of Y.

y	0	1	2	3	4
$p(y)$	0.2	0.25	0.3	0.2	0.05

since $p(0)$ must be such that the probabilities sum to 1.

(b) Find the variance of Y, showing your work.

$$E(Y) = (0)(0.2) + (1)(0.25) + (2)(0.30) + (3)(0.20) + (4)(0.05)$$

$$= .25 + .60 + .60 + .20 = 1.65$$

$$E(Y^2) = (0^2)(0.2) + (1^2)(0.25) + (2^2)(0.3) + (3^2)(0.2) + (4^2)(0.05) = .25 + 1.20 + 1.80 + .80 = 4.05$$

$$\Rightarrow V(Y) = 4.05 - (1.65)^2 = 1.3275$$

(c) If Coach Shane Beamer is paid a bonus of \$10000 for each home SEC game the team wins, find the expected value of his bonus. Show your work.

$$E(B) = \cancel{E(10Y)}$$

$$= E[10000\,Y] = 10000\,E(Y) = 10000(1.65)$$

$$= \$16,500$$

11. A botanist grows five specimens of a certain tree species, all in different greenhouses under identical conditions. Suppose the probability that each tree survives until reproductive maturity is 0.90, and that tree outcomes are independent.

(a) Find the expected value and variance of the number of trees that survive until maturity. Show how you got your answer. $Y \sim Bin(n=5, p=0.9)$

$$E(Y) = (5)(0.9) = 4.5$$

$$V(Y) = (5)(0.9)(0.1) = 0.45$$

(b) What is the probability that exactly three of the five trees survive until maturity? Indicate how you got your answer.

$$P(Y=3) = P(Y \le 3) - P(Y \le 2) = 0.081 - 0.009$$

$$= \boxed{0.072}$$

(c) What is the probability that FOUR OR MORE of the five trees survive until maturity? Indicate how you got your answer.

$$P(Y \ge 4) = 1 - P(Y \le 3)$$

$$= 1 - .081 = \boxed{0.919}$$