STAT 511/MATH 511 – Test 2 – Summer 2021 Key

1. Assume that the weights of newborn babies follow a normal distribution with mean 3.43 kg and standard deviation 0.48 kg. Professor Hitchcock has four kids (all boys). The weights of the first three at birth were 3.97 kg, 4.23 kg, and 4.65 kg.

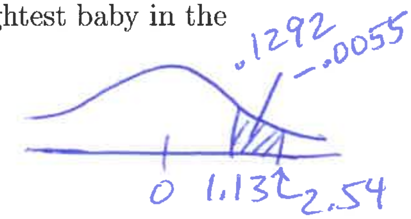
   (a) Assume we don't know the weight of the fourth baby yet. If that baby's weight follows the same distribution as the general population, then what is the probability that the fourth kid will be neither the heaviest nor the lightest baby in the family? Show work.

$$P(3.97 < Y < 4.65)$$

$$= P\left(\frac{3.97-3.43}{0.48} < Z < \frac{4.65-3.43}{0.48}\right)$$

$$= P(1.13 < Z < 2.54)$$

$$= [.1292 - .0055] = \boxed{.1237}$$

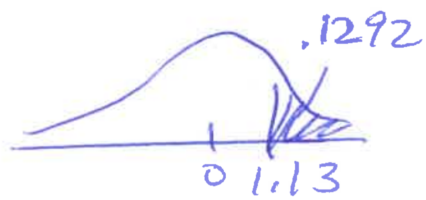


   (b) Spoiler alert: The fourth baby (now 7 years old) was in fact the heaviest of the four at birth. So what is the probability that four babies randomly selected from the general population will all weigh at least 3.97 kg? Show work.

$$P[Y > 3.97] = P\left[Z > \frac{3.97-3.43}{0.48}\right] = P[Z > 1.13]$$

$$= .1292$$

$$\Rightarrow (.1292)^4 = \boxed{.000279}$$



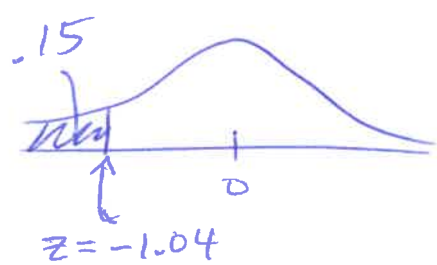
   (c) In the general population, suppose the lightest 15% of babies must be kept longer in the hospital for extra observation. What is the highest birthweight value (in kg) for which the baby will be kept for extra observation? Show work.

$$Z = -1.04$$

$$\frac{Y-3.43}{0.48} = -1.04$$

$$Y = (-1.04)(0.48) + 3.43$$

$$Y = \boxed{2.93 \text{ kg}}$$



2. Consider a random variable Y with a geometric distribution with $p = 0.3$. The moment generating function (mgf) of Y is denoted $m_Y(t)$. Let $W = 3Y + 5$.

(a) Show that the moment generating function of W is $m_W(t) = e^{5t}m_Y(3t)$.

$$m_w(t) = E[e^{tw}] = E[e^{t(3Y+5)}] = E[e^{5t}e^{3tY}]$$
$$= e^{5t}E[e^{(3t)Y}] = e^{5t}m_y(3t)$$

(b) Find $E(W)$ using known results about the geometric distribution and basic properties of expected values.

$$E(Y) = \frac{1}{0.3} = 3.333$$
$$E(W) = 3E(Y) + 5 = \boxed{15}$$

(c) Verify the answer for $E(W)$ by specifically using the mgf of W to derive $E(W)$.

$$m_w'(t) = [e^{5t}][3m_y'(3t)] + [5e^{5t}][m_y(3t)]$$
$$m_w'(0) = (1)[3m_y'(0)] + [5][m_y(0)]$$
$$= 3E(Y) + (5)1 = 3(3.333) + 5 = 15$$

3. Suppose the number of accidents per month at the intersection of Harden Street and Gervais Street follows a Poisson distribution with $\lambda = 3.6$.

(a) Find the probability that there will be 5 accidents in the next month. Show or briefly explain how you got your answer.

$$P(Y=5) = P(Y \leq 5) - P(Y \leq 4) = .844 - .706 = \boxed{.138}$$

or $$P(Y=5) = \frac{(3.6)^5 e^{-3.6}}{5!} = .138$$

(b) Find the probability that there will be two or fewer accidents in the next month.

$P(Y \leq 2) = .303$

(c) What is the expected number of accidents at the intersection in the next year?

$\lambda t = (3.6)(12) = 43.2$

(d) If there is exactly one accident in the next month, then what is the probability that single accident will occur in the last third of the month?

Time is Unif(0, 1) $\Rightarrow \frac{1}{3}$

4. Suppose Joseph K., who has been accused of a crime, goes to the courthouse each day to try to get his case heard by the judge. Assume that each day that chances that his case will be heard is 0.25, and that whether his case is heard is independent across all days.

(a) Find the probability that his case is finally heard on the fifth day that he goes. Show work.

$Y \sim \text{geometric}(p = 0.25)$

$P(Y=5) = (.25)(.75)^4 = 0.079$

(b) What is the expected number of days that he goes WITHOUT his case being heard (i.e., not counting the day on which it is heard)?

$E(Y-1) = E(Y) - 1 = \frac{1}{0.25} - 1 = 3$

(c) Suppose that once his case is heard, he then has to continue to return to the court each day to get the judge to announce his verdict. As before, the probability of seeing the judge each day is still 0.25, and days are independent. Assuming that the day on which his case is heard is unknown, what is the probability that his verdict is announced on the eighth day that he goes? Show work.

$Y \sim NB(r=2, p=0.25)$

$P(Y=8) = \binom{7}{1}(.25)^2(.75)^6 = \boxed{0.0779}$

5. The game of Scrabble involves randomly picking tiles with letters embossed on them from a bag. Suppose near the end of the game, there are 8 tiles in the bag: five are vowels and three are consonants.

   (a) A player will pick four tiles from the bag at random (without replacement). What is the probability that she picks three vowels and one consonant? Show work.

$Y \sim$ hypergeometric with $N=8$, $r=5$, $n=4$.

$$P(Y=3) = \frac{\binom{5}{3}\binom{3}{1}}{\binom{8}{4}} = \boxed{.4286}$$

   (b) Suppose, instead, that the player picked four tiles WITH replacement (each time noting whether the chosen tiles was a vowel or consonant and then putting it back in the bag). If in this experiment, Y is the total number of vowels out of the four chosen, what is the distribution of Y (including parameter values)?

Binomial $\left(n=4,\ p=\frac{5}{8}\right)$

6. Suppose Y is a random variable with the following moment generating function (mgf): $m_Y(t) = (0.5)e^t + (0.4)e^{3t} + (0.1)e^{5t}$.

   (a) Use the mgf directly to find $E(Y)$ and $V(Y)$. Show work.

$$m_y'(t) = 0.5e^t + 1.2e^{3t} + 0.5e^{5t}$$

$$E(Y) = m_y'(0) = 0.5 + 1.2 + 0.5 = \boxed{2.2}$$

$$m_y''(t) = 0.5e^t + 3.6e^{3t} + 2.5e^{5t}$$

$$E(Y^2) = m_y''(0) = 0.5 + 3.6 + 2.5 = 6.6$$

$$V(Y) = 6.6 - (2.2)^2 = \boxed{1.76}$$

   (b) Write the complete probability distribution of Y in the form of a table.

y	1	3	5
$p(y)$	0.5	0.4	0.1

7. Suppose that for a certain professional golfer playing a par-3 hole, the distance (in feet) that the golfer hits his ball from the hole follows a uniform distribution between 0 and 40 feet.

$Y \sim Unif(0, 40)$

(a) What is the expected distance that the golfer hits the ball from the hole?

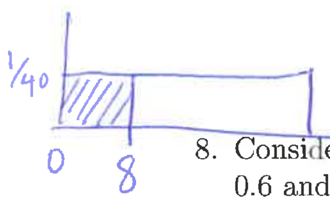
$$E(Y) = \frac{40+0}{2} = 20$$

(b) What is the *standard deviation* of the distance that the golfer hits the ball from the hole?

$$\sigma = \sqrt{V(Y)} = \sqrt{\frac{(40-0)^2}{12}} = \boxed{11.55}$$

(c) For putts of 8 feet or less, professional golfers are more likely to make the putt than to miss. What is the probability that the golfer hits his ball 8 feet or less from the hole?

$$P(Y<8) = P(0<Y<8) = \frac{8-0}{40-0} = .20$$



8. Consider a random variable Y that has the following probability function: $P(Y = 1) = 0.6$ and $P(Y = 3) = 0.4$.

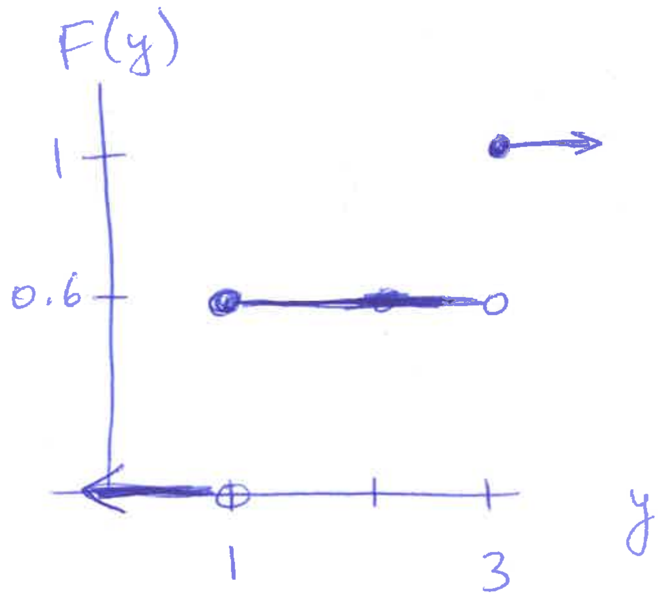
(a) Write the complete cumulative distribution function (cdf) of Y, and sketch a graph of the cdf $F(y)$.

If $y < 1$, $F(y) = 0$

If $y \geq 3$, $F(y) = 1$

If $1 \leq y < 3$, $F(y) = 0.6$

$$F(y) = \begin{cases} 0 & \text{if } y < 1 \\ 0.6 & \text{if } 1 \leq y < 3 \\ 1 & \text{if } y \geq 3 \end{cases}$$



(b) What is $F(2)$? What is $F(0.5)$?

$F(2) = 0.6 \qquad F(0.5) = 0$

9. Consider a continuous random variable Y with pdf of the form: $f(y) = ky^3$ for $0 < y < 2$ (and 0 elsewhere), where k is a constant.

(a) Show that the value of k that makes $f(y)$ a valid density is 0.25.

$$1 = \int_0^2 ky^3\,dy = k\left[\frac{y^4}{4}\right]_0^2 = k\left[\frac{16}{4} - 0\right]$$

$$\Rightarrow 4k = 1 \Rightarrow k = 0.25$$

(b) Derive a formula for the cdf $F(y)$ and write the complete cdf.

$$\int_0^y 0.25\,t^3\,dt = 0.25\left[\frac{t^4}{4}\right]_0^y = \frac{1}{16}y^4$$

$$F(y) = \begin{cases} 0 & \text{if } y < 0 \\ \frac{1}{16}y^4 & \text{if } 0 \le y \le 2 \\ 1 & \text{if } y > 2 \end{cases}$$

(c) Find both $E(Y)$ and the variance $V(Y)$, showing your work.

$$E(Y) = \int_0^2 y\left(0.25y^3\right)dy = 0.25\left[\frac{y^5}{5}\right]_0^2 = 0.25\left(\frac{32}{5}\right) = \boxed{1.6}$$

$$E(Y^2) = \int_0^2 y^2\left(0.25y^3\right)dy = 0.25\left[\frac{y^6}{6}\right]_0^2 = 0.25\left[\frac{64}{6} - 0\right]$$

$$= 2.667$$

$$V(Y) = 2.667 - (1.6)^2 = \boxed{0.10667}$$

(d) Find $P(Y > 1)$, showing your work.

$$P(Y>1) = 1 - P(Y \le 1) = 1 - \int_0^1 0.25 y^3 \, dy$$

$$= 1 - 0.25 \left[\frac{y^4}{4}\right]_0^1 = 1 - 0.25\left[\frac{1}{4} - 0\right] \cancel{\frac{15}{16}}$$

$$= 1 - \frac{1}{16} = \frac{15}{16} = \boxed{0.9375}$$

(e) Find $P(0.2 < Y < 1.5)$, showing your work.

$$\int_{0.2}^{1.5} 0.25 y^3 \, dy = 0.25 \left[\frac{y^4}{4}\right]_{0.2}^{1.5}$$

$$= 0.25 \left(\frac{1.5^4}{4} - \frac{0.2^4}{4} \right) = \boxed{0.3163}$$

(f) Show that the median (0.50 quantile) of the distribution of Y is approximately 1.6818.

$$\int_0^{1.6818} 0.25 y^3 \, dy = 0.25 \left[\frac{y^4}{4}\right]_0^{1.6818}$$

$$= 0.25 \left(\frac{1.6818^4}{4} - 0 \right) \approx 0.5$$

10. EXTRA CREDIT: If someone walks up to you on the street and says, "You appear to be someone who has a good grasp of mathematical statistics. Tell me, how do you show that the normal density function integrates to 1?" You would reply: "It's easy, you just ..." (fill in the rest of the sentence)

"... transform to polar coordinates!"