Stat509 Fall 2014 HW3 Solution

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Instruction: Please let me know if you find any error in this solution.

1. In 1990 the lead concentration in gasoline ranged from 0.1 to 0.5 grams/liter. Let Y = grams per liter of lead in gasoline. The probability density function for Y is

$$f(y) = 12.5y - 1.25, \text{ for } 0.1 < y < 0.5.$$

   (a) What is the probability that a random liter of gasoline would contain between 0.1 and 0.4 grams/liter of lead?
   Solution: $P(0.1 < Y < 0.4) = \int_{0.1}^{0.4} 12.5y - 1.25dy = 6.25y^2 - 1.25y|_{0.1}^{0.4} = 0.5625.$

   (b) What is the probability that a random liter of gasoline will contain more than 0.3 grams/liter of lead?
   Solution: $P(Y > 0.3) = \int_{0.3}^{0.5} 12.5y - 1.25dy = 6.25y^2 - 1.25y|_{0.3}^{0.5} = 0.75.$

   (c) Give the cumulative probability function $F_Y(y)$ (*Hint: you need to discuss the value of y by cases*).
   Solution: If $y < 0.1$, $F_Y(y) = P(Y \leq y) = 0$; If $0.1 \leq y \leq 0.5$, $F_Y(y) = P(Y \leq y) = \int_{0.1}^{y} 12.5t - 1.25dt = 6.25t^2 - 1.25t|_{0.1}^{y} = 6.25y^2 - 1.25y + 0.0625$; If $y > 0.5$, $F_Y(y) = P(Y \leq y) = 1$. In summary,

$$F_Y(y) = \begin{cases} 0 & \text{if } y < 0.1 \\ 6.25y^2 - 1.25y + 0.0625 & \text{if } 0.1 \leq y \leq 0.5 \\ 1 & \text{if } y > 0.5. \end{cases}$$

   (d) Use the cumulative probability function $F_Y(y)$ to calculate the probability that a random liter of gasoline will contain less than 0.35 grams of lead.
   Solution: By CDF derived above, $F_Y(0.35) = 6.25(0.35)^2 - 1.25(0.35) + 0.0625 = 0.3906.$

   (e) Calculate the expected value of Y.
   Solution: $E(Y) = \int_{-\infty}^{\infty} y f_Y(y)dy = \int_{0.1}^{0.5} y(12.5y - 1.25)dy = 0.3667.$

   (f) Calculate the variance for Y.
   Solution: $E(Y^2) = \int_{-\infty}^{\infty} y^2 f_Y(y)dy = \int_{0.1}^{0.5} y^2(12.5y - 1.25)dy = 0.1433$. So, $\text{Var}(Y) = E(Y^2) - [E(Y)]^2 = 0.0088.$

2. Explosive devices used in mining operations produce (nearly) circular craters when detonated. The radii of these craters measured in meters, say, Y, follow an exponential distribution with $\lambda = 0.10$.

   (a) Find the proportion of radii that will exceed 20 meters.
   Solution: $P(Y > 20) = e^{-\lambda 20} = e^{-0.1(20)} = 0.135.$

   (b) Find the probability that a single denotation will produce a radius between 5 and 15 meters.
   Solution: $P(5 < Y < 15) = F_Y(15) - F_Y(5) = (1 - e^{-0.1(15)}) - (1 - e^{-0.1(5)}) = e^{-0.1(5)} - e^{-0.1(15)} = 0.383.$

(c) The area of the crater is $W = \pi Y^2$. Find the expected (mean) area produced by the explosive devices; that is, compute $E(W)$.
Solution: $E(W) = E(\pi Y^2) = \pi E(Y^2) = \pi[\text{Var}(Y) + (E(Y))^2] = \pi[1/\lambda^2 + (1/\lambda)^2] = \pi(100 + 100) = 200\pi$.

3. Suppose X has an exponential distribution with a mean of 10. Calculate $P(X < 15|X > 10)$. (*Hint: apply the memoryless property*)
Solution: $E(X) = 1/\lambda = 10 \implies \lambda = 0.1$. $P(X < 15|X > 10) = P(X < 10 + 5|X > 10) = P(X < 5) = 1 - e^{-0.1(5)} = 0.393$.

4. The number of calls received by a telephone answering service follows a Poisson distribution. The calls average 20 per hour.

(a) What is the probability that 30 calls will arrive in a given 2 hour period?
Solution: Let X= number of calls received in 2 hours. So, the parameter is $\lambda t = 20(2) = 40$. So, $X \sim Pois(40)$. Therefore,

$$P(X = 30) = \frac{e^{-40}(40^{30})}{30!} = 0.0185.$$

(b) What is the probability of waiting more than 15 minutes between two calls? (*Hint: 15 minutes = 0.25 hour*)
Solution: Let Y = the waiting time between two calls, then $Y \sim exp(20)$. So,

$$P(X > 0.25) = e^{-20(0.25)} = 0.0067.$$

(c) What is the probability that there are 22 phone calls in a given hour?
Solution: Let W = number of calls received in a given hour, then $W \sim Pois(20)$. Therefore,

$$P(W = 22) = \frac{e^{-20}20^{22}}{22!} = 0.0769.$$

5. Suppose the weight, say, Y, in pounds of a certain packaged chemical is uniform from 48 to 50 pounds. That is the pdf is of the form

$$f_Y(y) = \frac{1}{2}, \text{ for } 48 < y < 50.$$

(a) What is the mean weight of the chemical?
Solution: $E(Y) = \int_{48}^{50} y/2dy = y^2/4|_{48}^{50} = 49$.

(b) What is the probability that a randomly chosen package of chemical will weigh between 48.5 and 49.4 pounds?
Solution: $P(48.5 < Y < 49.4) = \int_{48.5}^{49.4} 1/2dy = 1/2(49.4 - 48.5) = 0.45$.

(c) In the long run, what proportion of packages will weigh more than 49.2 pounds?
Solution: Recall that we can interpret probability as long-run proportion, so we want to calculate

$$P(Y > 49.2) = \int_{49.2}^{50} 1/2dy = (50 - 49.2)/2 = 0.4.$$