Stat509 Fall 2014 Homework 4 Solution

Instructor: Peijie Hou

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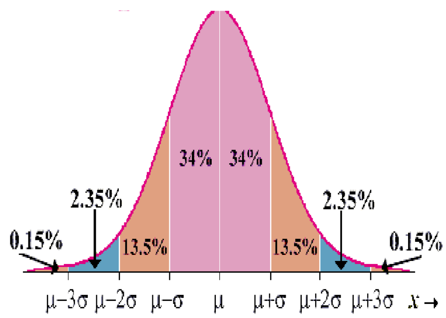
Instruction: Let me know if you find any error on this solution.

1. Suppose Y is a normal random variable with a mean of 25 and a standard deviation of 1.5. Apply Empirical rule, calculate the following probability.

   (a) Approximately what proportion of the distribution lies between 23.5 and 26.5?
   Solution: This is equivalent to find the probability of the distribution lies within 1 standard deviation from the mean, since $|23.5 - 25| = |26.5 - 25| = 1.5 = \sigma$. By Empirical rule, the probability is 68%.

   (b) Approximately what proportion of the distribution lies between 23.5 and 28?
   Solution: Another way to view Empirical rule is via the following plot. Let $X \sim N(25, 1.5^2)$,



   $$P(23.5 < X < 28) = P(\mu - \sigma < X < \mu + 2\sigma) = 0.34 + 0.34 + 0.135 = 0.815.$$

   (c) Approximately what proportion of the distribution is greater than 28?
   Solution: $P(X > 28) = 0.0235 + 0.0015 = 0.025$.

2. The weight of a ream of paper follows a normal distribution with a mean of 2.8 pounds and a standard deviation of 0.8 pounds. Use R or standard normal table to answer the following questions

   (a) Approximately what proportion of the reams weigh less than 2.2 pounds?
   Solution: Let X denotes the weight of a randomly chosen ream.

   $$P(X < 2.2) = P\left(\frac{X - 2.8}{0.8} < \frac{2.2 - 2.8}{0.8}\right) = P(Z < -0.75) = 0.2266.$$

   In R,

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> pnorm(2.2,2.8,0.8)
[1] 0.2266274
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(b) Find out the value in pound such that 5% of the reams weigh more than this value.
Solution: We want to find a constant c, such that $P(X > c) = 0.05 \iff P(X \le c) = 0.95$. Apply the standard normal transformation,

$$P(X < c) = P(Z < \frac{c - 2.8}{0.8}) = 0.95.$$

From the table, $(c - 2.8)/0.8 = 1.64$ or 1.65, it doesn't matter which one you choose, let's pick 1.64 as an example, $c = 1.64(0.8) + 2.2 = 4.112$. We conclude that about 5% of the ream is greater than 4.112 pounds. In R,

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> qnorm(0.95,2.8,0.8)
[1] 4.115883
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(c) What is the probability that a randomly chosen ream will weigh less than 0.9392 pounds?
Solution: $P(X < 0.9392) = P(Z < (0.9392 - 2.8)/0.8) = P(Z < -2.33) = 0.0099$. In R,

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> pnorm(0.9392,2.8,0.8)
[1] 0.01000928
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(d) What is the approximate probability that a randomly chosen ream will weigh between 1.5 and 3.5 pounds?
Solution:

$$\begin{aligned} P(1.5 < X < 3.5) &= P(X < 3.5) - P(X < 1.5) \\ &= P(Z < (3.5 - 2.8)/0.8) - P(Z < (1.5 - 2.8)/0.8) \\ &= P(Z < 0.875) - P(Z < -1.625) \\ &= 0.757 \end{aligned}$$

In R,

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> pnorm(3.5,2.8,0.8)-pnorm(1.5,2.8,0.8)
[1] 0.7571318
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(e) If the specification calls for the reams to weigh 2.8 ± 1.6 pounds, about what proportion of the reams are out-of-specification?
Solution: $P(\text{out-of-specification}) = 1 - P(\text{within-specification}) = 1 - P(2.8 - 1.6 < X < 2.8 + 1.6) = 1 - P(1.2 < X < 4.4) = 0.0455$. This is similar to the above question. In R,

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> 1-(pnorm(4.4,2.8,0.8)-pnorm(1.2,2.8,0.8))
[1] 0.04550026
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3. Assume that in the detection of a digital signal the background noise follows a normal distribution with a mean of 0 volt and standard deviation of 0.45 volt. The system assumes a digital 1 has been transmitted when the voltage exceeds 0.9.

(a) What is the probability of detecting a digital 1 when none was sent?
If there is no signal sent, and the system still detects a digital 1, which means the noise is

greater than 0.9 volt. Let $N =$ the noise reading, then $N \sim \mathcal{N}(0, 0.45)$.

$P(\text{detecting a digital 1 when none was sent}) = P(N > 0.9) = P(Z > 0.9/0.45) = P(Z > 2) = 0.025$

by Empirical rule. Note that such probability is also called probability of false detection.

(b) Determine symmetric bounds about 0 that include 99% of all noise readings.
Solution: We want to find a $x_0 > 0$ such that $P(-x_0 < X < x_0) = 0.99$. It follows that $P(X < x_0) = 1 - 0.005 = 0.995$. Therefore,

$$P(X < x_0) = P(Z < x_0/0.45) = 0.995.$$

From the normal probability table, $x_0/0.45 = 2.57 \implies x_0 = 2.57(0.45) = 1.1565$.

4. (Redo questions in notes) The army reports that the distribution of head circumferences among male soldiers is approximately normal with a mean of 22.8 inches and standard deviation of 1.1 inch. The army plans to make helmets in advance to fit the middle 98% of head circumferences for male soldiers. What head circumferences are small enough or big enough to require custom fitting?
Solution: We first find the value of z_0 such that $P(-z_0 < Z < z_0) = 0.98$. From standard normal table, $z_0 = 2.33$. That is, the maximum helmet z-score is 2.33 and the minimum helmet z-score is -2.33. Their corresponding “inches” are 25.363 and 20.237, respectively. Hence, the small enough size is 20.237 and big enough size is 25.363.

5. (Redo questions in notes) The distribution of head circumferences among female soldiers is approximately normal with mean 22.2 inches and standard deviation 1.4 inches. Female soldiers use the same type of helmet as male soldiers (from previous question). What percent of female soldiers can be fitted with a made-in-advance helmet?
Solution: The made-in-advance helmet can fit from 20.237 to 25.363 (from previous question). Therefore,

$$P(20.237 < X < 25.363) = P(-1.40 < Z < 2.26) = 0.907.$$

In R,

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> pnorm(25.363,22.2,1.4)-pnorm(20.237,22.2,1.4)
[1] 0.9076309
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