STAT509: Continuous Random Variable

Peijie Hou

University of South Carolina

September 23, 2014

Continuous Random Variable

- A continuous random variable is a random variable with an interval (either finite or infinite) of real numbers for its range.
- Examples
  - Let Y = length in meter.
  - Let Y = temperature in $^\circ$F.
  - Let Y = time in seconds

Continuous Random Variable Cont'd

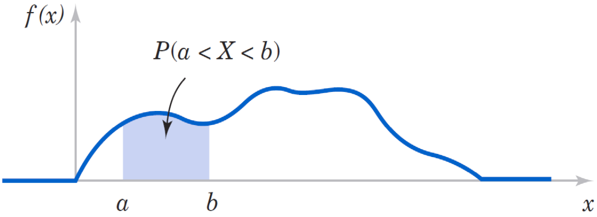
- Because the number of possible values of X is uncountably infinite, the probability mass function (pmf) is no longer suitable.
- For a continuous random variable, $P(Y = y) = 0$, the reason for that will become clear shortly.
- For a continuous random variable, we are interested in probabilities of intervals, such as $P(a \leq Y \leq b)$, where a and b are real numbers.
- We will introduce the probability density function to calculate probabilities, such as $P(a \leq Y \leq b)$.

Probability Density Function

- Every **continuous random variable** X we will cover in STAT509 has a **probability density function (pdf)**, denoted by $f_X(x)$.
- Probability density function $f_X(x)$ is a function such that
  a. $f_X(x) \geq 0$ for any $x \in \mathbb{R}$
  b. $\int_{-\infty}^{\infty} f_X(x)dx = 1$
  c. $P(a \leq X \leq b) = \int_a^b f_X(x)dx =$ area under $f_X(x)$ from a to b for any $b > a$.
  d. If x_0 is a specific value, then $P(X = x_0) = 0$. We assign 0 to area under a point.

Cumulative Distribution Function

- Here is a pictorial illustration of pdf:



- Let X_0 be a specific value of interest, the **cumulative distribution function (CDF)** is defined via

$$F_X(x_0) = P(X \leq x_0) = \int_{-\infty}^{x_0} f_X(x)dx.$$

Cumulative Distribution Function Cont'd

- If x_1 and x_2 are specific values, then

$$
\begin{aligned}
P(x_1 \leq X \leq x_2) &= \int_{x_1}^{x_2} f_X(x)dx \\
&= F_X(x_2) - F_X(x_1).
\end{aligned}
$$

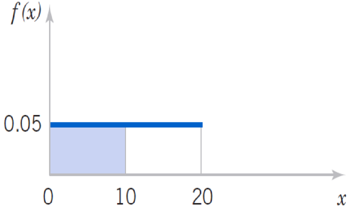
- From last property of a pdf, we have

$$P(x_1 \leq X \leq x_2) = P(x_1 < X < x_2)$$

Example: Electric Current

Let the continuous random variable X denote the current measured in a thin copper wire in milliamperes. Assume that the range of X (measured in mA) is [0, 20], and assume that the probability density function of X is $f_X(x) = 0.05$ for $0 \leq x \leq 20$. for What is the probability that a current measurement is less than 10 milliamperes?

Solution: The plot of pdf of X is



Example: Electric Current Cont'd

Example: CDF Calculation

Suppose that Y has the pdf

$$f_Y(y) = \begin{cases} 3y^2, & 0 < y < 1 \\ 0, & \text{otherwise.} \end{cases}$$

Solution:

Mean of a Continuous R.V.

- The mean and variance can also be defined for a continuous random variable. Integration replaces summation in the discrete definitions.
- Recall that for a **discrete** random variable Y. The mean of Y is defined as

$$\mathrm{E}(Y) = \mu_Y = \sum_{\text{all } y} y \cdot p_Y(y).$$

- Definition: For a **continuous** random variable X. The mean of X is defined as

$$\mathrm{E}(X) = \mu_X = \int_{-\infty}^{\infty} x f_X(x) dx$$

Mean of a Continuous R.V. Cont'd

- NOTE: The limits of the integral in this definition, while technically correct, will always be the lower and upper limits corresponding to the nonzero part of the pdf.

- Definition: Just like in the discrete case, we can calculate the expected value for a function of a continuous r.v. Let X be a continuous random variable with pdf $f_X(x)$. Suppose that g is a real-valued function. Then, $g(X)$ is a random variable and

$$\mathrm{E}\left[g(X)\right] = \int_{-\infty}^{\infty} g(x) f_X(x) dx.$$

Variance of a Continuous R.V.

- Definition: The variance of X, denoted as $\mathrm{Var}(X)$ or σ^2, is

$$\sigma^2 = \mathrm{Var}(X) = \mathrm{E}(X-\mu)^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f_X(x)\, dx.$$

- The population standard deviation of X is

$$\sigma = \sqrt{\sigma^2},$$

  the positive square root of the variance.
- The computational formula for variance is the same as the discrete case, i.e.,

$$\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2.$$

Example: Electric Current

- Recall that the pdf of X is

$$f_X(x) = \begin{cases} 0.05, & 0 < x < 20 \\ 0, & \text{otherwise.} \end{cases}$$

- Let's calculate the mean of X:

Example: Electric Current Cont'd

What about the variance of X?

Introduction to Exponential Distribution

- We have discussed Poisson distribution in the previous chapter, which, for example, can model the number of car accidents for a given length of time t in certain intersection.
- The *waiting time* between accidents is another random variable that is often of interest. We can use exponential distribution to model such a waiting period.
- In general, let the random variable N denote the number of accidents during time of length x, and X be the waiting time. If the mean number of accidents is λ per base unit, then r.v. N follows a Poisson distribution with mean λx.

Introduction to Exponential Distribution

- To model the waiting time, suppose there is no accident during the time of length x. Now,

$$P(X > x) = P(N = 0) = \frac{e^{-\lambda x}(\lambda x)^0}{0!} = e^{-\lambda x}.$$

- It follows that

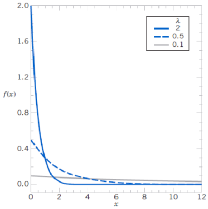
$$F_X(x) = P(X \le x) = 1 - P(X > x) = 1 - e^{-\lambda x}.$$

- By differentiating the CDF of X, the pdf of X is

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \ge 0 \\ 0, & \text{otherwise.} \end{cases}$$

Exponential Distribution

- The plot of pdf of exponential distribution with differenct values of λ is shown below:



- The shorthand notation for X following exponential distribution is given by

$$X \sim \exp(\lambda)$$

Mean and Variance for an Exponential Random Variable

- Suppose that $X \sim \exp(\lambda)$, then $E(X) =$

- The variance is

- Note, for exponential r.v., mean $=$ standard deviation.

Summary on exponential distribution

Suppose that $X \sim \exp(\lambda)$, then

- pdf: $f_X(x) = \lambda e^{-\lambda x}$, for $x \geq 0$
- CDF: $F_X(x) = P(X \leq x) = 1 - e^{-\lambda x}$, for $x \geq 0$

  Note: $P(X > x) = 1 - F_X(x) = e^{-\lambda x}$, for $x \geq 0$
- Mean: $\mathrm{E}(X) = \dfrac{1}{\lambda}$
- Variance: $\mathrm{Var}(X) = \dfrac{1}{\lambda^2}$

Example: Computer Usage

Let X denote the time in hours from the start of the interval until the first log-on. Then, X has an exponential distribution with 1 log-ons per hour. We are interested in the probability that X exceeds 6 minutes.
(*Hint: Because λ is given in log-ons per hour, we express all time units in hours. That is, 6 minutes =0.1 hour*)

Solution:

Example: Accidents

The time between accidents at a factory follows an exponential distribution with a historical average of 1 accident every 900 days. What is the probability that there will be more than 1200 days between the next two accidents?

Solution:

Example: Accidents Cont'd

If the time between accidents follows an exponential distribution with a mean of 900 days, what is the probability that there will be less than 900 days between the next two accidents?

Solution:

Exponential or Poisson Distribution?

- We model the number of industrial accidents occurring in one year.
- We model the length of time between two industrial accidents (assuming an accident occurring is a Poisson event).
- We model the time between radioactive particles passing by a counter (assuming a particle passing by is a Poisson event).
- We model the number of radioactive particles passing by a counter in one hour

Example: Radioactive Particles

- The arrival of radioactive particles at a counter are Poisson events. So the number of particles in an interval of time follows a Poisson distribution. Suppose we average 2 particles per millisecond
- What is the probability that no particles will pass the counter in the next 3 milliseconds?
- What is the probability that more than 3 milliseconds will elapse before the next particle passes?

More Example: Machine Failures

- If the number of machine failures in a given interval of time follows a Poisson distribution with an average of 1 failure per 1000 hours, what is the probability that there will be no failures during the next 2000 hours?
  Solution:

- What is the probability that the time until the next failure is more than 2000 hours?
  Solution:

It's Your Turn...

Number of failures in an interval of time follows a Poisson distribution. If the mean time to failure is 250 hours, what is the probability that more than 2000 hours will pass before the next failure occurs?

a e^{-8}

b $1 - e^{-8}$

c $e^{-\frac{1}{8}}$

d $1 - e^{-\frac{1}{8}}$

Lack of Memory Property

- An even more interesting property of an exponential random variable is concerned with conditional probabilities.

- The exponential distribution is often used in reliability studies as the model for the time until failure of a device. The lack of memory property of the exponential distribution implies that the device does not wear out, i.e., $P(L < t + \Delta t | L > t)$ remains the same for any t.

- However, the lifetime L of a device that suffers slow mechanical wear, such as bearing wear, is better modeled by a distribution s.t. $P(L < t + \Delta t | L > t)$ increases with t, such as the **Weibull distribution** (later).

Proof of Lack of Memory Property

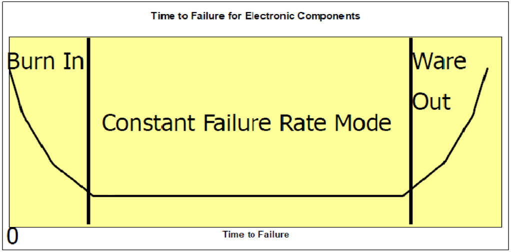
Proof:

Understanding Lack of Memory Property

To understand this property of exponential distribution, let us assume X models the life time of a light bulb. The lack of memory property tells you that given the fact that the light bulb still "survives" at time t_1, the probability it will work less than additional t_2 amount of time (the conditional probability) equals to the probability that it will work less than t_2 amount of time from the beginning (unconditional probability).

Common Application for the Exponential Dist.

- Mathematically, it can be shown that the exponential distribution is the **only** continuous probability distribution that has a constant failure rate due to its memoryless property.
- Time between failures when a product is in constant failure rate mode.



Constant Failure Rate Mode

What is the probability that an electrical component in constant failure rate mode with an average time to failure of 2000 hours will still be operating at 2500 hours?

Solution:

Constant Failure Rate Mode Cont'd

If ten of these components are placed in different devices, what is the probability that at least one will still be operating at 2500 hours?

Solution:

Constant Failure Rate Mode Cont'd

For how many hours would you guarantee an electrical component that is in constant failure rate mode if the average time to failure is 5000 hours and you want no more than 5% of the components subject to the guarantee? (*Hint:* $\ln(e^x) = x$)

Solution:

Introduction to Weibull Distribution

- As mentioned previously, the Weibull distribution is often used to model the time until failure of many different physical systems.

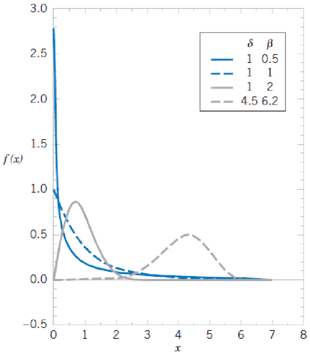
- The random variable X with probability density function

$$f(x) = \frac{\beta}{\delta}\left(\frac{x}{\delta}\right)^{\beta-1} e^{-(x/\delta)^{\beta}}, \text{ for } x > 0$$

is a **Weibull random variable** with scale parameter $\delta > 0$ and shape parameter $\beta > 0$.

Introduction to Weibull Distribution Cont'd

The parameters in the distribution provide a great deal of flexibility to model systems in which the number of failures increases with time (bearing wear), decreases with time (some semiconductors), or remains constant with time (failures caused by external shocks to the system).



CDF of Weibull Distribution Cont'd

If X has a Weibull distribution with parameters δ and β , then the cumulative distribution function of X is

$$F(x) = 1 - e^{-(x/\delta)^{\beta}}.$$

Example: Bearing Wear

The time to failure (in hours) of a bearing in a mechanical shaft is satisfactorily modeled as a Weibull random variable with $\beta = 1/2$ and $\delta = 5000$ hours. Determine the probability that a bearing lasts at least 6000 hours.

Solution:

Introduction to Normal Distribution

- Most widely used model.
- Central Limit Theorem (more later): Whenever a random experiment is replicated, the random variable that equals the average (or total) result over the replicates tends to have a normal distribution as the number of replicates becomes large.
- Other names: **Gaussian distribution**, "bell-shaped distribution" or "bell-shaped curve."

Density of Normal Distribution

- A random variable X with probability density function

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty$$

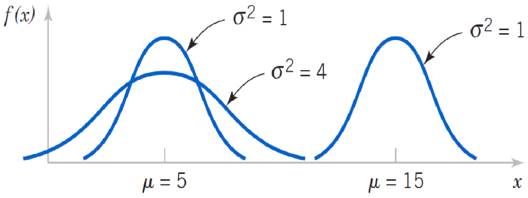
  is a normal random variable with parameters μ and σ, where $-\infty < \mu < \infty$, and $\sigma > 0$. Also,

$$\mathrm{E}(X) = \mu \quad \text{and} \quad \mathrm{Var}(X) = \sigma^2.$$

- We use $X \sim \mathcal{N}(\mu, \sigma^2)$ to denote the distribution.
- Our objective now is to calculate probabilities (of intervals) for a normal random variable through R or normal probability table.

Density of Normal Distribution Cont'd

- The plot of normal distribution with different parameter values:

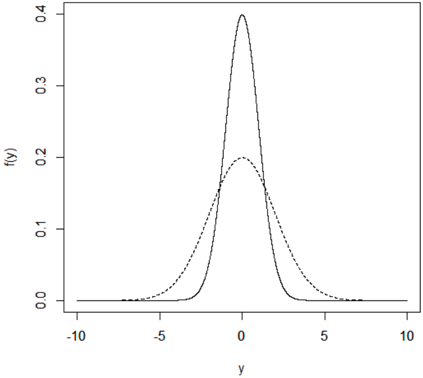


- CDF: The cdf of a normal random variable does not exist in closed form. Probabilities involving normal random variables and normal quantiles can be computed numerically.

Characteristics of Normal Pdf

- Bell-shaped curve
- $-\infty < x < \infty$, i.e., the range of X is the whole real line
- μ determines distribution location and is the highest point on curve
- Curve is symmetric about μ
- σ determines distribution spread

It's Your Turn...



Which pdf in the above has greater variance?

Example: Strip of Wire

Assume that the current measurements in a strip of wire follow a normal distribution with a mean of 10 milliamperes and a variance of 4 (milliamperes)^2. What is the probability that a measurement exceeds 13 milliamperes?

Solution: Let X denotes the measure in that strip of wire. Then, $X \sim \mathcal{N}(10, 4)$. We want to calculate

$$P(X > 13) = 1 - P(X \leq 13).$$

```
> 1-pnorm(13,10,2)
[1] 0.0668072
```

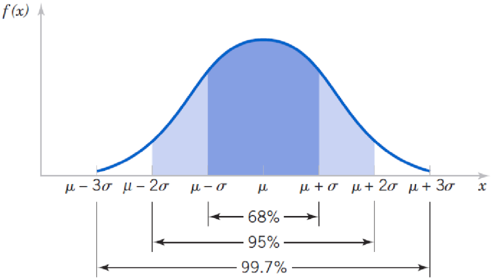
Note code of calculating $F(x) = P(X \leq x)$ is of the form $\texttt{pnorm}(\texttt{x}, \mu, \sigma)$.

Empirical Rule

For any normal random variable X,

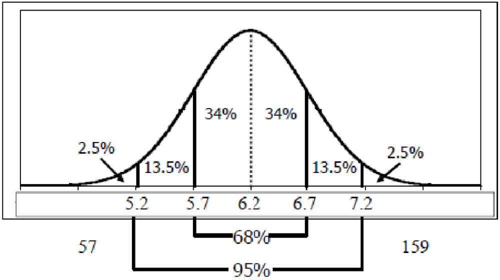
$$P(\mu - \sigma < X < \mu + \sigma) = 0.6827$$
$$P(\mu - 2\sigma < X < \mu + 2\sigma) = 0.9543$$
$$P(\mu - 3\sigma < X < \mu + 3\sigma) = 0.9973$$

These are summarized in the following plot:



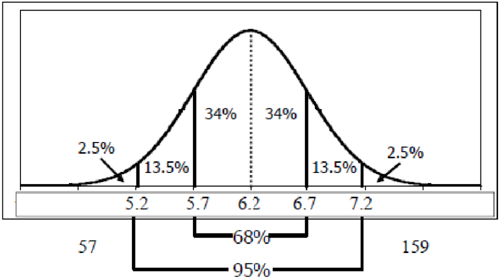
Earthquakes in a California Town

- Since 1900, the magnitude of earthquakes that measure 0.1 or higher on the Richter Scale in a certain location in California is distributed approximately normally, with $\mu = 6.2$ and $\sigma = 0.5$, according to data obtained from the United States Geological Survey.
- Earthquake Richter Scale Readings



Earthquakes in a California Town Cont'd

- Approximately what percent of the earthquakes are above 5.7 on the Richter Scale?
- What is the highest an earthquake can read and still be in the lowest 2.5%?
- What is the approximate probability an earthquake is above 6.7?



Standard Normal Distribution

- If X is a normal random variable with $\mathrm{E}(X) = \mu$ and $\mathrm{Var}(X) = \sigma^2$, the random variable

$$Z = \frac{X - \mu}{\sigma}$$

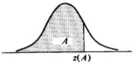
  is a normal random variable with $\mathrm{E}(Z) = 0$ and $\mathrm{Var}(Z) = 1$. That is, Z is a standard normal random variable.

- Creating a new random variable by this transformation is referred to as standardizing.

- Z is traditionally used as the symbol for a standard normal random variable.

Normal Probability Table

Entry is area A under the standard normal curve from $-\infty$ to $z(A)$



z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

Normal Probability Table Cont'd

- With the help of normal probability table, we can calculate the probabilities for nonstandard normal distribution through **standardizing**.
- Suppose $X \sim \mathcal{N}(10, 4)$, we want to calculate $P(X > 13)$.

$$\begin{aligned} P(X > 13) &= P(\frac{X-10}{\sqrt{4}} > \frac{13-10}{\sqrt{4}}) \\ &= P(Z > 1.5) \\ &= 1 - P(Z \leq 1.5) \\ &= 1 - 0.9332 \text{ (from table)} \\ &= 0.0668 \end{aligned}$$

Normal Probability Table Cont'd

- If we want to calculate $P(X < 7)$,

$$
\begin{aligned}
P(X < 7) &= P(\frac{X - 10}{\sqrt{4}} < \frac{7 - 10}{\sqrt{4}}) \\
&= P(Z < -1.5) \\
&= 0.0668
\end{aligned}
$$

- If we want to calculate $P(X > 7)$,

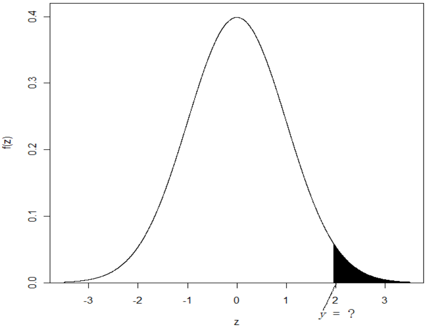
$$
\begin{aligned}
P(X > 7) &= P(\frac{X - 10}{\sqrt{4}} > \frac{7 - 10}{\sqrt{4}}) \\
&= P(Z > -1.5) \\
&= P(Z < 1.5) \text{ (by symmetry)} \\
&= 0.9332
\end{aligned}
$$

Example: Steel Bolt

- The thickness of a certain steel bolt that continuously feeds a manufacturing process is normally distributed with a mean of 10.0 mm and standard deviation of 0.3 mm. Manufacturing becomes concerned about the process if the bolts get thicker than 10.5 mm or thinner than 9.5 mm.
- Find the probability that the thickness of a randomly selected bolt is greater than 10.5 or smaller than 9.5 mm.
  Solution:

Inverse Normal Probabilities

Sometimes we want to answer a question which is the reverse situation. We know the probability, and want to find the corresponding value of Y.



Inverse Normal Probabilities Cont'd

What is the cutoff value that approximately 2.5% of the bolts produced will have thicknesses less than this value?

Solution: We need to find the z value such that $P(Z < z) = 0.025$. We can transform back to the original cutoff value from z. The R code of finding z is

```
> qnorm(0.025)
[1] -1.959964
```

Or, $z = -1.96$ by table. It follows that

$$
\begin{aligned}
P(Z < -1.96) = 0.025 \quad &\Longleftrightarrow \quad P\left(\frac{X - 10}{0.3} < -1.96\right) = 0.025 \\
&\Longleftrightarrow \quad P(X < 9.412) = 0.025
\end{aligned}
$$

Therefore, the cutoff value is 9.412.

Inverse Normal Probabilities Cont'd

What is the cutoff value that approximately 1% of the bolts produced will have thicknesses greater than this value?

Solution:

Normal Distribution Exercises

The fill volume of an automatic filling machine used for filling cans of carbonated beverage is normally distributed with a mean of 12.4 fluid ounces and a standard deviation of 0.1 fluid ounce.

- What is the probability that a randomly chosen can will contain between 12.3 and 12.5 ounces?
  Solution:
- 2.5% of the cans will contain less than ___ ounces.
  Solution:

Normal Distribution Exercises Cont'd

The mean of the filling operation can be adjusted easily, but the standard deviation remains at 0.1 ounce.

- At what value should the mean be set so that 97.5% of all cans exceed 12.0 ounces?
  Solution:
- At what value should the mean be set so that 97.5% of all cans exceed 12.0 ounces if the standard deviation can be reduced to 0.05 ounces?
  Solution:

Normal Distribution Exercises Cont'd

The army reports that the distribution of head circumferences among male soldiers is approximately normal with a mean of 22.8 inches and standard deviation of 1.1 inch. The army plans to make helmets in advance to fit the middle 98% of head circumferences for male soldiers. What head circumferences are small enough or big enough to require custom fitting?

Solution:

The distribution of head circumferences among female soldiers is approximately normal with mean 22.2 inches and standard deviation 1.4 inches. Female soldiers use the same type of helmet as male soldiers. What percent of female soldiers can be fitted with a made-in-advance helmet?

Solution: