Solutions for Stat 512 — Take home exam I

1. Let Y has an exponential distribution with mean β. Prove that $W = \sqrt{Y}$ has a Weibull density. (Hint: Using the C.D.F technique) (20 pts)

Solution:

$$Y \sim Exp(\beta) \qquad \Longrightarrow \qquad f_Y(y) = \frac{1}{\beta} exp(-\frac{y}{\beta}),\ y \geq 0$$

Hence,

$$\begin{aligned} F_W(w) = P(W \leq w) &= P\left(\sqrt{Y} \leq w\right) \\ &= P(0 < Y \leq w^2) \\ &= \int_0^{w^2} \frac{1}{\beta} exp(-\frac{y}{\beta}) dy \\ &= -exp(-\frac{y}{\beta})\Big|_{y=0}^{w^2} \\ &= 1 - exp(-\frac{w^2}{\beta}), \qquad w \geq 0 \end{aligned}$$

Furthermore,

$$f_W(w) = \frac{dF_W(w)}{dw} = \frac{2w}{\beta} exp(-\frac{w^2}{\beta}), \qquad w\ \geq 0$$

From Wikipedia, the pdf of Weibull distribution is :

$$f_Y(y) = \frac{k}{\lambda}\left(\frac{y}{\lambda}\right)^{k-1} e^{-(y/\lambda)^k}, y \geq 0$$

Hence, the pdf of W is Weibull distribution with $k = 2$ and $\lambda = \sqrt{\beta}$.

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2. Let $Y_1, Y_2 \sim N(0, 1)$, Y_1, Y_2 are independent random variable. Find the distribution of $U = \dfrac{Y_1}{Y_2}$. (Hint: Using transformation technique) (20 pts)

Solution:

$$\begin{aligned} & Y_1, Y_2 \overset{ind.}{\sim} N(0,1) \\ \Longrightarrow\ & f_{Y_1,Y_2}(y_1, y_2) = f_{Y_1}(y_1) f_{Y_2}(y_2) = \frac{1}{2\pi} exp\left(-\frac{y_1^2 + y_2^2}{2}\right), \qquad y_1, y_2 \in (-\infty, \infty) \end{aligned}$$

For part (a), let $U = \dfrac{Y_1}{Y_2}$ and $V = Y_2$. Hence,

$$\begin{cases} Y_1 = h_1^{-1}(U,V) = UV \\ Y_2 = h_2^{-1}(U,V) = V \end{cases} \implies J = \begin{vmatrix} V & U \\ 0 & 1 \end{vmatrix} = V$$

How about the support?
If $V > 0$, then $-\infty < U < \infty$ and

$$\begin{aligned} f_{U,V}(u,v) &= f_{Y_1,Y_2}(h_1^{-1}, h_2^{-1})|J| \\ &= \frac{v}{2\pi} exp\left(-\frac{u^2v^2+v^2}{2}\right), \qquad v \in (0,\infty), u \in (-\infty,\infty) \end{aligned}$$

If $V < 0$, then $-\infty < U < \infty$ and

$$\begin{aligned} f_{U,V}(u,v) &= f_{Y_1,Y_2}(h_1^{-1}, h_2^{-1})|J| \\ &= -\frac{v}{2\pi} exp\left(-\frac{u^2v^2+v^2}{2}\right), \qquad v \in (-\infty,0), u \in (-\infty,\infty) \end{aligned}$$

If $V = 0$, then U is undefined. (But do we care about the case $V = 0$?)

Hence,

$$f_U(u) = \int_0^{\infty} \frac{v}{2\pi} exp\left(-\frac{u^2v^2+v^2}{2}\right) dv + \int_{-\infty}^{0} -\frac{v}{2\pi} exp\left(-\frac{u^2v^2+v^2}{2}\right) dv$$

For the first integral, let $w = v^2$, then

$$\begin{aligned} \int_0^{\infty} \frac{v}{2\pi} exp\left(-\frac{u^2v^2+v^2}{2}\right) dv &= \int_0^{\infty} \frac{1}{4\pi} exp\left(-\frac{w(u^2+1)}{2}\right) dw \\ &= \frac{1}{4\pi}\left(-\frac{2}{u^2+1}\right) exp\left(-\frac{w}{2/(u^2+1)}\right)\Bigg|_0^{\infty} \\ &= \frac{1}{2\pi}\frac{1}{u^2+1}, u \in (-\infty,\infty) \end{aligned}$$

For the second integral, let $w = v^2$, then

$$\begin{aligned} \int_{-\infty}^{0} -\frac{v}{2\pi} exp\left(-\frac{u^2v^2+v^2}{2}\right) dv &= \int_{\infty}^{0} -\frac{1}{4\pi} exp\left(-\frac{w(u^2+1)}{2}\right) dw \\ &= -\frac{1}{4\pi}\left(-\frac{2}{u^2+1}\right) exp\left(-\frac{w}{2/(u^2+1)}\right)\Bigg|_{\infty}^{0} \\ &= \frac{1}{2\pi}\frac{1}{u^2+1}, u \in (-\infty,\infty) \end{aligned}$$

Therefore,

$$f_U(u) = \frac{1}{2\pi}\frac{1}{u^2+1} + \frac{1}{2\pi}\frac{1}{u^2+1} = \frac{1}{\pi}\frac{1}{u^2+1}, u \in (-\infty, \infty)$$

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3. Suppose that a unit of mineral ore contains a proportion Y_1 of metal A and a Proportion Y_2 of metal B. Experience has shown that the joint probability density function of Y_1 and Y_2 is uniform over the region $0 \le y_1 \le 1, 0 \le y_2 \le 1, 0 \le y_1 + y_2 \le 1$. Let $U = Y_1 + Y_2$, the proportion of either metal A or B per unit. Find

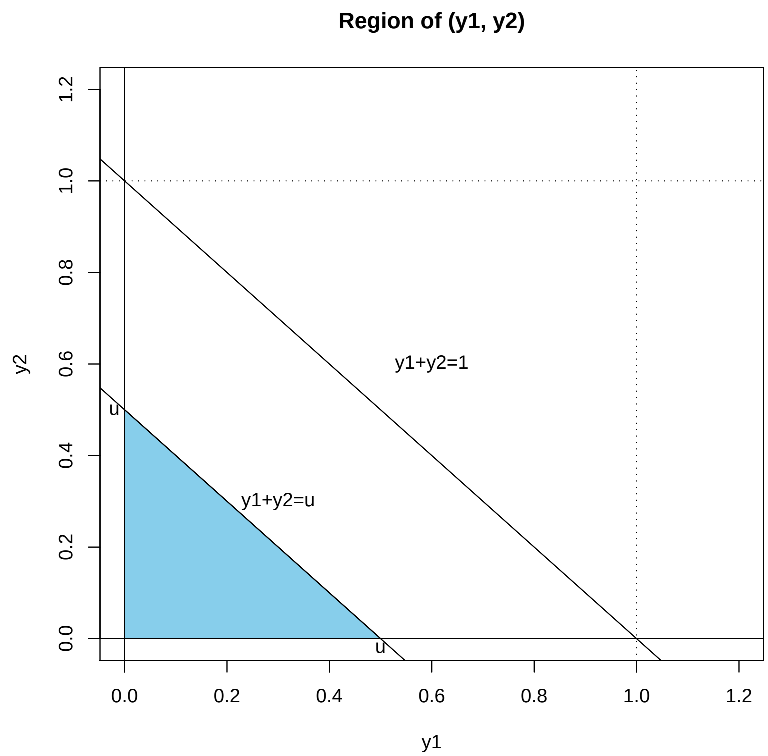
a) the probability density function for U. (10 pts)

b) $E(U)$ and $Var(U)$ by using the answer to part (a). (10 pts)

Solution:
For part (a): Firstly we should find out the joint density of (Y_1, Y_2), it is shown as below:

$$f_{Y_1,Y_2}(y_1, y_2) = 2 \qquad if\ 0 \le y_1, y_2 \le 1,\ 0 \le y_1 + y_2 \le 1$$

Notice: The joint density is not 1 since we have an additional constrain : $0 \le y_1 + y_2 \le 1$.
The region of (Y_1, Y_2) that satisfies the condition $U \le u$ is shown below:



Hence,

$$\begin{aligned} F_U(u) &= P(U \leq u) \\ &= P(Y_1 + Y_2 \leq u) \\ &= \int_0^u \int_0^{u-y_1} 2dy_2 dy_1 \\ &= u^2, \qquad u \in [0,1] \end{aligned}$$

Otherwise $F_U(u) = 0$ for $u \in (-\infty, 0)$ and 1 for $u \in (1, \infty)$. Also, $f_U(u) = \frac{dF_U(u)}{du} = 2u$ for $u \in [0,1]$.

For part (b),

$$\begin{aligned} E(U) &= \int_0^1 u \cdot 2udu = \frac{2}{3} \\ E(U^2) &= \int_0^1 u^2 \cdot 2udu = \frac{1}{2} \end{aligned}$$

Hence, $E(U) = \frac{2}{3}$ and $Var(U) = E(U^2) - [E(U)]^2 = \frac{1}{18}$.

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4. The total time from arrival to completion of service at a fast-food outlet , Y_1 and the time spent waiting in line before arriving at the service window, Y_2, have a joint density function:

$$f(y_1, y_2) = \begin{cases} e^{-y_1}, & 0 \le y_2 \le y_1 < \infty \\ 0, & o.w \end{cases}$$

Another random variable of interest is $U = Y_1 - Y_2$, the time spent at the service window. Find:

a) the probability density function for U using bivariate transformation technique. (10 pts)

b) $E(U)$ and $Var(U)$ by using the answer to part (a). (10 pts)

Solution:
For part (a), we can set up another r.v V such that $V = Y_1$. Hence,

$$\begin{cases} Y_1 = V \\ Y_2 = V - U \end{cases} \implies J = \begin{vmatrix} 0 & 1 \\ -1 & 1 \end{vmatrix} = 1$$

It is not hard to identify the support of (U, V) is: $U \ge 0, \ , V \ge U$. Hence,

$$f_{U,V}(u, v) = e^{-v} \cdot |1| = e^{-v}, \qquad u \ge 0, \ , v \ge u$$

And,

$$f_U(u) = \int_u^\infty e^{-v} dv = e^{-u}, \qquad u \ge 0$$

Hence, the time spent at the service window, U, has standard Exponential distribution.
For part (b), from the property of Exponential distribution, the mean and variance for U are all 1.

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5. Let $Y_1, \ldots, Y_n$ be i.i.d random variables such that for $0 < p < 1$, $P(Y_i = 1) = p$ and $P(Y_i = 0) = q = 1 - p$. (Such random variables are called Bernoulli random variables.)

a. Find the moment generating function for the Bernoulli random variable Y_1. Make sure you show your steps. (10pts)

b. Find the moment generating function for $W = Y_1 + Y_2 + \ldots + Y_n$. Can you recognize which known distribution has this mgf? (10 pts)

Solution:

a. For Bernoulli distribution:

$$\begin{aligned} m_{y_1}(t) &= E(e^{ty_1}) \\ &= \sum_{y_1=0 \ or \ 1} e^{ty_1} p^{y_1} (1-p)^{1-y_1} \\ &= (1-p) + pe^t \end{aligned}$$

b. Since $Y_1, \ldots, Y_n$ are i.i.d,

$$\begin{aligned} m_W(t) &= [m_{y_1}(t)]^n \\ &= \left[(1-p) + pe^t\right]^n \end{aligned}$$

Hence, the distribution of W is Binomial(n, p).

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Extra credit question: If $Y_i, i = 1, 2$, are independent Gamma$(\alpha_i, 1)$ random variables, find the marginal distributions of $U_1 = \dfrac{Y_1}{Y_1 + Y_2}$ and $U_2 = \dfrac{Y_2}{Y_1 + Y_2}$.

Solution:
It is easier to find the distribution of U_1 and U_2 separately. For U_1,

$$Let \left\{ \begin{array}{l} U = \dfrac{Y_1}{Y_1 + Y_2} \\ V = Y_1 + Y_2 \end{array} \right. \implies \left\{ \begin{array}{l} Y_1 = UV \\ Y_2 = V(1-U) \end{array} \right. \implies J = V > 0 \qquad (since\ gamma)$$

For the support:

$$\left\{ \begin{array}{l} Y_1 = UV > 0 \\ Y_2 = V(1-U) > 0 \end{array} \right. \implies \left\{ \begin{array}{l} V > 0 \\ 0 < U < 1 \end{array} \right.$$

Hence,

$$\begin{aligned} f_{U,V}(u, v) &= \frac{1}{\Gamma(\alpha_1)} (uv)^{\alpha_1 - 1} e^{-uv} \frac{1}{\Gamma(\alpha_2)} (v(1-u))^{\alpha_2 - 1} e^{-v(1-u)} \cdot v \\ &= \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} u^{\alpha_1 - 1} (1-u)^{\alpha_2 - 1} v^{\alpha_1 + \alpha_2 - 1} e^{-v} \end{aligned}$$

From above joint density, it is straightforward to see that U and V are independent. Also the kernel of U is $u^{\alpha_1 - 1}(1-u)^{\alpha_2 - 1}$ which is the kernel for $Beta(\alpha_1, \alpha_2)$. Hence, $U_1 = \dfrac{Y_1}{Y_1 + Y_2}$ has $Beta(\alpha_1, \alpha_2)$ distribution. (You can also integrate the joint density w.r.t v to get the marginal density of U). Similarly, U_2 has $Beta(\alpha_2, \alpha_1)$ distribution.