

Quiz 3 — Solutions

STAT 515 · Normal Distributions, Exponential, Sampling Distributions (Homework 5 material)

Total Points: 40

Problem 1: Short Answer (8 points)

(a) (4 points) *Central Limit Theorem.* If $X_1, \dots, X_n$ is a **random sample** (independent observations) from a population with mean μ and finite standard deviation σ , then for **sufficiently large** n the sampling distribution of $\bar{X}$ is approximately normal with

$$\mu_{\bar{X}} = \mu, \quad \sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}}.$$

The approximation improves as n increases; $n \geq 30$ is the usual working rule.

Grading: 2 points for the random-sample / large- n assumptions, 2 points for the normal conclusion with the correct mean and standard error.

(b) (4 points) Points curving away from the reference line at **both** ends indicate that the tails of the data are heavier (or lighter) than a normal distribution's — the distribution is not normal in the tails even though the center looks normal. Upward curvature at both ends means heavy tails / outliers in both directions. I would **not** treat the data as normal; a normality-dependent procedure (for example a χ^2 interval for σ^2) should not be used.

Grading: 2 points for identifying tail behavior (heavy tails / outliers, not skewness), 2 points for the correct conclusion.

Problem 2: The Normal Distribution (12 points)

$X \sim N(\mu = 72, \sigma = 8)$.

(a) (4 points)

$$z = \frac{84 - 72}{8} = 1.50, \quad P(X > 84) = P(Z > 1.50) = 1 - 0.9332 = \boxed{0.0668}$$

Grading: 2 points for the correct z , 2 points for the probability.

(b) (4 points)

$$z_1 = \frac{64 - 72}{8} = -1.00, \quad z_2 = \frac{80 - 72}{8} = 1.00$$

$$P(64 < X < 80) = P(-1 < Z < 1) = 0.8413 - 0.1587 = \boxed{0.6826}$$

(This is the Empirical Rule's 68%.)

Grading: 2 points for both z -scores, 2 points for the probability.

(c) (4 points) The 90th percentile has $z = 1.28$:

$$x = \mu + z\sigma = 72 + 1.28(8) = 72 + 10.24 = \boxed{82.24 \text{ bpm}}$$

Grading: 2 points for $z = 1.28$, 2 points for the back-transformation. (Using $z = 1.2816$ gives 82.25; accept.)

Problem 3: Sampling Distribution of $\bar{X}$ (12 points)

$\mu = 15$, $\sigma = 6$, $n = 49$.

(a) (4 points)

$$\mu_{\bar{X}} = \mu = 15 \text{ minutes}, \quad \sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}} = \frac{6}{\sqrt{49}} = \frac{6}{7} = \boxed{0.857 \text{ minutes}}$$

By the Central Limit Theorem, because the sample is random and $n = 49 \geq 30$ is large, $\bar{X}$ is approximately normal **regardless of the skewed shape of the population**. The CLT describes the distribution of the sample mean, not of individual observations.

Grading: 1 point mean, 1 point standard error, 2 points for invoking the CLT and the large- n condition.

(b) (4 points)

$$z = \frac{16.5 - 15}{0.857} = 1.75, \quad P(\bar{X} > 16.5) = P(Z > 1.75) = 1 - 0.9599 = \boxed{0.0401}$$

Grading: 1 point for using $\sigma_{\bar{X}}$ rather than σ , 1 point for the correct z , 2 points for the probability.
Common error: dividing by $\sigma = 6$ instead of 0.857 (gives $z = 0.25$) — at most 1 point.

(c) (4 points)

$$\sigma_{\bar{X}} = \frac{6}{\sqrt{196}} = \frac{6}{14} = \boxed{0.4286 \text{ minutes}}$$

The sample size increased by a factor of **4**; the standard error decreased by a factor of **2**. In general, to cut the standard error in half you must **quadruple** the sample size, because $\sigma_{\bar{X}}$ depends on $\sqrt{n}$.

Grading: 2 points for the new SE, 1 point for the two factors, 1 point for the $\sqrt{n}$ explanation.

Problem 4: The Exponential Distribution (8 points)

(a) (4 points) Y is **exponential** with mean $\theta = 1/\lambda = 1/4$ hour = 15 minutes.

For an exponential distribution the standard deviation equals the mean, so $\sigma_Y =$ 15 minutes as well.

Grading: 1 point for naming the exponential distribution, 2 points for $\theta = 15$ minutes, 1 point for the standard deviation.

(b) (4 points)

$$P(Y \geq 20) = e^{-20/15} = e^{-1.3333} = \text{0.2636}$$

Grading: 2 points for the correct formula with $a = 20$ and $\theta = 15$ in the same units, 2 points for the value. **Common error:** mixing units ($a = 20$ minutes with $\theta = 0.25$ hours).

R verification

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# Problem 2
1 - pnorm(84, 72, 8)           # 0.0668
pnorm(80, 72, 8) - pnorm(64, 72, 8) # 0.6827
qnorm(0.90, 72, 8)           # 82.25

# Problem 3
se <- 6 / sqrt(49); se        # 0.8571
1 - pnorm(16.5, 15, se)      # 0.0401
6 / sqrt(196)                # 0.4286

# Problem 4
pexp(20, rate = 1/15, lower.tail = FALSE) # 0.2636
```