

HOMEWORK 3 — SOLUTIONS

Probability

Problem 1: Die and Coin Experiment (20 points)

Part (a)

The sample space S consists of all (die outcome, coin outcome) pairs:

$$S = \{(1,H),(2,H),(3,H),(4,H),(5,H),(6,H),(1,T),(2,T),(3,T),(4,T),(5,T),(6,T)\}$$

$|S| = 12$ equally likely outcomes

Part (b)

Define the events:

$$\begin{aligned} A = \text{"even die"} &= \{(2,H),(4,H),(6,H),(2,T),(4,T),(6,T)\} \\ B = \text{"Heads"} &= \{(1,H),(2,H),(3,H),(4,H),(5,H),(6,H)\} \\ A \cap B &= \{(2,H),(4,H),(6,H)\} \\ A \cup B &= \{(1,H),(2,H),(3,H),(4,H),(5,H),(6,H),(2,T),(4,T),(6,T)\} \end{aligned}$$

Part (c)

$$\begin{aligned} P(A) &= 6/12 = 1/2 \\ P(B) &= 6/12 = 1/2 \\ P(A \cap B) &= 3/12 = 1/4 \\ P(A \cup B) &= P(A) + P(B) - P(A \cap B) = 1/2 + 1/2 - 1/4 = 3/4 \end{aligned}$$

Mutually exclusive? NO — $A \cap B \neq \emptyset$ (they share outcomes).

Independent? YES — $P(A \cap B) = 1/4 = P(A) \cdot P(B) = (1/2)(1/2)$. ✓

Part (d)

$$\begin{aligned} C = \text{"die"} > 4 &= \{(5,H),(6,H),(5,T),(6,T)\}, \quad P(C) = 4/12 = 1/3 \\ A \cap C &= \{(6,H),(6,T)\}, \quad P(A \cap C) = 2/12 = 1/6 \\ P(A | C) &= P(A \cap C)/P(C) = (1/6)/(1/3) = 1/2 \end{aligned}$$

Since $P(A | C) = 1/2 = P(A)$, events A and C are independent. ✓

(Knowing the die shows > 4 does not change the probability it is even.)

Problem 2: Student Health Survey (20 points)

Part (a)

Construct the 2×2 contingency table (counts):

	D (diet)	D ^c (no diet)
E (exercise)	60	60
E ^c (no exercise)	30	50

Totals: $E \rightarrow 120$, $E^c \rightarrow 80$; $D \rightarrow 90$, $D^c \rightarrow 110$; Grand total: 200. Cells: $E \cap D = 60$, $E \cap D^c = 60$, $E^c \cap D = 30$, $E^c \cap D^c = 50$.

Part (b)

$$\begin{aligned} P(E \cup D) &= P(E) + P(D) - P(E \cap D) \\ &= 120/200 + 90/200 - 60/200 \\ &= 0.60 + 0.45 - 0.30 = 0.75 \end{aligned}$$

Interpretation: 75% of students either exercise regularly, eat a balanced diet, or both.

Part (c)

$$P(E | D) = P(E \cap D) / P(D) = (60/200) / (90/200) = 60/90 \approx 0.667$$

Interpretation: Among students who eat a balanced diet, about 66.7% also exercise regularly.

Part (d)

For independence, we need $P(E \cap D) = P(E) \cdot P(D)$.

$$P(E) \cdot P(D) = 0.60 \times 0.45 = 0.27$$

$$P(E \cap D) = 60/200 = 0.30$$

Since $0.30 \neq 0.27$, events E and D are NOT independent.

Students who exercise are more likely to also eat well (positive association).

Problem 3: Software Quality Control (10 points)

Part (a)

Let A, B, C = project from Team A, B, C.

$$P(A) = 0.50, P(B) = 0.30, P(C) = 0.20$$

$$P(\text{Bug}|A) = 0.04, P(\text{Bug}|B) = 0.06, P(\text{Bug}|C) = 0.03$$

Total Probability Rule:

$$P(\text{Bug}) = P(\text{Bug}|A) \cdot P(A) + P(\text{Bug}|B) \cdot P(B) + P(\text{Bug}|C) \cdot P(C)$$

$$= 0.04 \times 0.50 + 0.06 \times 0.30 + 0.03 \times 0.20$$

$$= 0.020 + 0.018 + 0.006 = 0.044$$

Part (b)

$$P(B | \text{Bug}) = P(\text{Bug} | B) \cdot P(B) / P(\text{Bug})$$

$$= (0.06 \times 0.30) / 0.044$$

$$= 0.018 / 0.044 \approx 0.4091$$

If a project has a critical bug, there is about a 40.9% chance it came from Team B.

Problem 4: Fundamental Theorem of Counting (15 points)

Password structure: [Upper(1)] [Digit(3)] [Lower(2)]

Part (a)

Apply the Fundamental Theorem of Counting — multiply independent choices:

$$\text{Total} = 26 \times 10 \times 10 \times 10 \times 26 \times 26$$

$$= 26^3 \times 10^3$$

$$= 17,576 \times 1,000$$

$$= 17,576,000 \text{ valid passwords}$$

Part (b)

Digits must be distinct (no repetition among the 3 digit positions):

First digit: 10 choices

Second digit: 9 choices (one digit used)

Third digit: 8 choices (two digits used)

$$\text{Total} = 26 \times 10 \times 9 \times 8 \times 26 \times 26$$

$$= 26^3 \times 720$$

$$= 17,576 \times 720$$

$$= 12,654,720 \text{ valid passwords}$$

Part (c)

Fix first character = "A" (1 way) and last character = "z" (1 way); all others free:

$$\text{Favorable} = 1 \times 10 \times 10 \times 10 \times 26 \times 1 = 26,000$$

$$\text{Total} = 17,576,000 \text{ (from part a)}$$

$$P = 26,000 / 17,576,000 = 26 / 17,576 = 1 / 676 \approx 0.001479$$

Equivalently: $P = (1/26) \times 1 \times 1 \times 1 \times (1/26) \times (1/26) \dots$

No — last TWO chars are lowercase; only the very last is fixed to "z".

$$P = P(\text{first} = A) \times P(\text{last} = z) = (1/26) \times (1/26) = 1/676 \approx 0.00148$$

Problem 5: Combinations (20 points)

Part (a)

$$\begin{aligned} C(52, 5) &= 52! / (5! \times 47!) \\ &= (52 \times 51 \times 50 \times 49 \times 48) / (5 \times 4 \times 3 \times 2 \times 1) \\ &= 311,875,200 / 120 \\ &= 2,598,960 \text{ possible 5-card hands} \end{aligned}$$

Part (b)

Exactly 3 Aces: choose 3 of the 4 Aces, then 2 non-Aces from the remaining 48 cards:

$$\begin{aligned} C(4, 3) \times C(48, 2) \\ &= 4 \times [48 \times 47 / 2] \\ &= 4 \times 1,128 \\ &= 4,512 \text{ hands} \end{aligned}$$

Part (c)

$$\begin{aligned} P(\text{exactly 3 Aces}) &= C(4,3) \times C(48,2) / C(52,5) \\ &= 4,512 / 2,598,960 \\ &\approx 0.001736 \text{ (about 0.17\%)} \end{aligned}$$

Part (d)

A flush: all 5 cards from the same suit. Each suit has 13 cards:

$$\begin{aligned} \text{Flushes per suit} &= C(13, 5) = 13! / (5! \times 8!) = 1,287 \\ \text{Total flushes} &= 4 \times 1,287 = 5,148 \end{aligned}$$

(This count includes straight flushes and royal flushes, as stated in the problem.)