

HOMEWORK 4 — SOLUTION KEY

Binomial, Poisson, Continuous Random Variables (§4.6–4.9)

Problem 1: Binomial Distribution ($n = 10, p = 0.3$)

X = number of correct guesses, $X \sim \text{Binomial}(n = 10, p = 0.3, q = 0.7)$

(a) $P(X = 4)$:

$$\begin{aligned} P(X = 4) &= C(10, 4) \cdot (0.3)^4 \cdot (0.7)^6 \\ &= 210 \times 0.0081 \times 0.117649 \\ &= 210 \times 0.00095296 \\ &\approx 0.2001 \end{aligned}$$

(b) $P(X \geq 2)$ via complement:

$$\begin{aligned} P(X \geq 2) &= 1 - P(X \leq 1) = 1 - [P(X = 0) + P(X = 1)] \\ P(X = 0) &= C(10, 0) \cdot (0.3)^0 \cdot (0.7)^{10} = (0.7)^{10} = 0.028248 \\ P(X = 1) &= C(10, 1) \cdot (0.3)^1 \cdot (0.7)^9 = 10 \times 0.3 \times 0.040354 = 0.121061 \\ P(X \leq 1) &= 0.028248 + 0.121061 = 0.149309 \\ P(X \geq 2) &= 1 - 0.149309 \approx 0.8507 \end{aligned}$$

(c) $E(X)$ and $\text{Var}(X)$:

$$\begin{aligned} E(X) &= np = 10 \times 0.3 = 3 \\ \text{Var}(X) &= npq = 10 \times 0.3 \times 0.7 = 2.1 \\ \text{Interpretation:} &\text{ On average, 3 out of 10 questions are guessed correctly.} \end{aligned}$$

Problem 2: Poisson Distribution ($\lambda = 4$ per 10 min)

X = number of customer arrivals in 10 minutes, $X \sim \text{Poisson}(\lambda = 4)$

$$P(X = x) = e^{-4} \cdot 4^x / x! \quad e^{-4} \approx 0.018316$$

(a) $P(\text{exactly 3 arrivals})$:

$$\begin{aligned} P(X = 3) &= e^{-4} \cdot 4^3 / 3! \\ &= 0.018316 \times 64 / 6 \\ &= 0.018316 \times 10.6667 \\ &\approx 0.1954 \end{aligned}$$

(b) $P(\text{at most 2 arrivals})$:

$$P(X = 0) = e^{-4} \cdot 4^0 / 0! = 0.018316$$

$$P(X = 1) = e^{-4} \cdot 4^1 / 1! = 0.018316 \times 4 = 0.073263$$

$$P(X = 2) = e^{-4} \cdot 4^2 / 2! = 0.018316 \times 8 = 0.146525$$

$$P(X \leq 2) = 0.018316 + 0.073263 + 0.146525 = 0.238104 \approx 0.2381$$

(c) Mean and variance:

$$E(X) = \lambda = 4 \quad \text{Var}(X) = \lambda = 4$$

A key property of the Poisson distribution: mean equals variance.

Problem 3: Continuous Random Variable $f(x) = 2x$ on $[0, 1]$

(a) Verify $f(x)$ is a valid density:

(i) Non-negativity: $f(x) = 2x \geq 0$ for all $x \in [0, 1]$. ✓

(ii) Integrates to 1:

$$\int_0^1 2x \, dx = [x^2]_0^1 = 1^2 - 0^2 = 1 \quad \checkmark$$

Both conditions are satisfied — $f(x)$ is a valid probability density function.

(b) $P(0.25 \leq X \leq 0.75)$:

$$\begin{aligned} P(0.25 \leq X \leq 0.75) &= \int_{0.25}^{0.75} 2x \, dx \\ &= [x^2]_{0.25}^{0.75} \\ &= (0.75)^2 - (0.25)^2 \\ &= 0.5625 - 0.0625 \\ &= 0.5000 \end{aligned}$$

There is a 50% probability that X falls in the middle half of $[0, 1]$.

(c) $E(X)$:

$$\begin{aligned} E(X) &= \int_0^1 x \cdot f(x) \, dx = \int_0^1 x \cdot 2x \, dx = \int_0^1 2x^2 \, dx \\ &= [2x^3 / 3]_0^1 \\ &= 2(1)^3 / 3 - 0 \\ &= 2/3 \approx 0.6667 \end{aligned}$$

Problem 4: Binomial — Insurance Fraud ($n = 15$, $p = 0.08$)

X = number of fraudulent claims, $X \sim \text{Binomial}(n = 15, p = 0.08, q = 0.92)$

$P(X \leq 1) = P(X = 0) + P(X = 1)$:

$$P(X = 0) = C(15, 0) \cdot (0.08)^0 \cdot (0.92)^{15} = (0.92)^{15} \approx 0.2863$$

$$P(X = 1) = C(15, 1) \cdot (0.08)^1 \cdot (0.92)^{14}$$

$$\begin{aligned}
 &= 15 \times 0.08 \times (0.92)^{14} \\
 &= 1.2 \times 0.311193 \\
 &\approx 0.3734
 \end{aligned}$$

$$P(X \leq 1) = 0.2863 + 0.3734 = 0.6597$$

Interpretation: There is about a 65.97% chance that at most 1 of the 15 claims is fraudulent.

Problem 5: R Code Verification for Problem 4

The R command to verify $P(X \leq 1)$ for $\text{Binomial}(n=15, p=0.08)$:

```
# Problem 4 verification in R
pbinom(1, size = 15, prob = 0.08)
# Output: [1] 0.6597232

# Can also compute individual probabilities:
dbinom(0, size = 15, prob = 0.08) # P(X = 0) ≈ 0.2863
dbinom(1, size = 15, prob = 0.08) # P(X = 1) ≈ 0.3734

# Sum should match pbinom result:
dbinom(0, 15, 0.08) + dbinom(1, 15, 0.08) # ≈ 0.6597
```

R output: $\text{pbinom}(1, \text{size}=15, \text{prob}=0.08) = 0.6597$
 Hand calculation: 0.6597 ✓
 Results are consistent.

Problem 6: Poisson R Commands: dpois and ppois ($\lambda = 5$)

(a) What each command returns:

`dpois(2, lambda=5)`: Returns the probability mass at exactly $x = 2$.
This is $P(X = 2)$ — the probability that exactly 2 events occur when the average rate is $\lambda = 5$.

`ppois(2, lambda=5)`: Returns the cumulative probability up to $x = 2$.
This is $P(X \leq 2) = P(X=0) + P(X=1) + P(X=2)$.
`ppois` is the running sum that includes all values from 0 through x .

(b) Hand calculation ($e^{-5} \approx 0.006738$):

`dpois(0, lambda=5)`: $P(X=0) = e^{-5} \cdot 5^0/0! = 0.006738 \times 1 = 0.006738$
`dpois(1, lambda=5)`: $P(X=1) = e^{-5} \cdot 5^1/1! = 0.006738 \times 5 = 0.033690$
`dpois(2, lambda=5)`: $P(X=2) = e^{-5} \cdot 5^2/2! = 0.006738 \times 12.5 = 0.084224$

`ppois(2, lambda=5)` = $P(X=0) + P(X=1) + P(X=2)$
= $0.006738 + 0.033690 + 0.084224$
= $0.124652 \approx 0.1247$

```
# Verification in R:
dpois(2, lambda=5) # [1] 0.08422434
ppois(2, lambda=5) # [1] 0.1246520

# Confirm ppois = sum of dpois(0), dpois(1), dpois(2):
sum(dpois(0:2, lambda=5)) # [1] 0.1246520 ✓
```

`ppois(2,  $\lambda=5$ )` = `dpois(0, $\lambda=5$ )` + `dpois(1, $\lambda=5$ )` + `dpois(2, $\lambda=5$ )`
 $0.1247 = 0.0067 + 0.0337 + 0.0842$ ✓ (consistent)

Problem 7: Normal Distribution and the Empirical Rule

$X \sim N(\mu = 70 \text{ inches}, \sigma = 3 \text{ inches})$

(a) Distribution description and labeled values:

The distribution is bell-shaped and symmetric about $\mu = 70$.

Key labeled values on the horizontal axis:

$$\mu - 3\sigma = 61 \quad \mu - 2\sigma = 64 \quad \mu - \sigma = 67$$

$$\mu = 70 \text{ (center)}$$

$$\mu + \sigma = 73 \quad \mu + 2\sigma = 76 \quad \mu + 3\sigma = 79$$

The curve peaks at 70 and tapers symmetrically toward both tails.

(b) $P(67 < X < 73)$ — within 1 standard deviation:

$$67 = \mu - \sigma = 70 - 3 \quad \text{and} \quad 73 = \mu + \sigma = 70 + 3$$

Applying the 68–95–99.7 Rule ($\pm 1\sigma$ interval):

$$P(67 < X < 73) = P(\mu - \sigma < X < \mu + \sigma) \approx 0.68$$

About 68% of adult men have heights between 67 and 73 inches.

(c) $P(64 < X < 76)$ — within 2 standard deviations:

$$64 = \mu - 2\sigma = 70 - 6 \quad \text{and} \quad 76 = \mu + 2\sigma = 70 + 6$$

Applying the 68–95–99.7 Rule ($\pm 2\sigma$ interval):

$$P(64 < X < 76) = P(\mu - 2\sigma < X < \mu + 2\sigma) \approx 0.95$$

About 95% of adult men in this population have heights between 64 and 76 inches.

(d) $P(X > 76)$ using symmetry:

Step 1: $P(64 < X < 76) \approx 0.95$ (from part c)

Step 2: Area outside $\pm 2\sigma = 1 - 0.95 = 0.05$

Step 3: By symmetry of the Normal, both tails are equal:

$$P(X < 64) = P(X > 76) = 0.05 / 2 = 0.025$$

$$P(X > 76) \approx 0.025 \quad (2.5\% \text{ of men are taller than 76 inches})$$

(e) Is 61 inches unusual?

Number of standard deviations from the mean:

$$z = (x - \mu) / \sigma = (61 - 70) / 3 = -9 / 3 = -3$$

61 inches is exactly 3 standard deviations below the mean.

By the Empirical Rule: 99.7% of values fall within $\pm 3\sigma$,
so only 0.3% fall outside. By symmetry, $P(X < 61) \approx 0.15\%$.

Conclusion: Yes — a height of 61 inches is highly unusual for
this population (it falls in the extreme lower 0.15% of the distribution).