Homework 6 — Solutions

STAT 515 · The CLT in Practice and Confidence Intervals for a Mean · 75 points

Instructor copy.

Problem 1: Conditions for the Central Limit Theorem (12 points)

(a) Central Limit Theorem. Let $X_1, \ldots, X_n$ be a random sample from a population with mean μ and finite standard deviation σ.

- *Population:* any shape — it need not be normal — but μ and σ must exist and be finite.
- *Sample:* the observations must be **independent** and drawn at random from that population.
- *Conclusion:* for sufficiently large n, $\bar{X} \mathrel{\dot\sim} N\big(\mu,\ (\sigma/\sqrt{n})^2\big)$, and the approximation improves as n grows.

In this course "sufficiently large" means roughly $n > 25$–30; the more skewed the population, the larger n must be. If the population is *already* normal, $\bar{X}$ is exactly normal for every n.

(b)

1. **No.** The independence condition fails. There are only 10 students, each measured 4 times; the 4 commutes belonging to one student are correlated with one another. The effective sample size is closer to 10 than to 40.
2. **No.** The population is strongly right-skewed and $n = 12$ is far below the rule of thumb, so the CLT has not had a chance to take effect. The sampling distribution of $\bar{X}$ will still be noticeably right-skewed.
3. **Yes.** Same skewed population, but $n = 200$ is large, so the CLT applies and $\bar{X}$ is approximately normal.
4. **Approximately normal, yes — but note the sampling fraction.** $n = 50$ is large enough for the CLT to deliver the normal shape. However, $50/300 = 16.7\%$ of the population, which **violates the usual 10% guideline** for treating draws without replacement as independent. Sampling without replacement from a finite population makes the observations slightly dependent, and the true standard error is *smaller* than $\sigma/\sqrt{n}$ by the finite-population correction $\sqrt{(N-n)/(N-1)}$. So the normal approximation is fine and using $\sigma/\sqrt{n}$ is conservative — it overstates the uncertainty a little. *(Award full credit for identifying that* $50/300 > 10\%$*; the direction of the correction is a bonus.)*

(c) Two commute times from the *same* student tend to be alike — a student who lives far away has a long commute every day. So the 40 measurements carry less information than 40 commutes from 40 different students: the repeated values are partly redundant. Because the observations are

positively correlated, the true standard error of $\bar{X}$ is **larger** than $\sigma/\sqrt{40}$, and using $\sigma/\sqrt{40}$ would make the estimate look more precise than it really is.

Problem 2: Confidence Interval for a Mean, σ Known (15 points)

$\sigma = 1.2$ (known), $n = 36$, $\bar{x} = 14.8$.

(a)

$$\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}} = \frac{1.2}{\sqrt{36}} = \frac{1.2}{6} = 0.20 \text{ hours.}$$

(b) 95% interval, $z^* = 1.96$:

$$14.8 \pm 1.96(0.20) = 14.8 \pm 0.392 \implies (14.408,\ 15.192) \text{ hours.}$$

(c) 99% interval, $z^* = 2.576$:

$$14.8 \pm 2.576(0.20) = 14.8 \pm 0.515 \implies (14.285,\ 15.315) \text{ hours.}$$

(d) The 99% interval is wider (width 1.03 vs. 0.78). To be more confident that the interval captures μ, you must cast a wider net. With the data fixed, confidence and precision trade directly against each other; the only way to gain both is to collect more data.

(e) No. Knowing σ removes the need for the t distribution, but it does nothing about the *shape* of the sampling distribution of $\bar{X}$. The interval still relies on $\bar{X}$ being approximately normal, which requires either a normal population or a large enough n for the CLT. Here $n = 36$ is large enough, so the interval is fine — but for the right reason, not the one she gave.

Problem 3: Interpreting a Confidence Interval (15 points)

(a) This confuses an interval for the **mean** with an interval for **individuals**. The CI describes where the population *average* μ plausibly lies; it says nothing about the spread of individual people's sodium intake. Individual intakes vary far more widely than a 270-mg window — many adults fall well outside it.

(b) μ is a **fixed, unknown constant**, not a random variable. Once the sample is drawn and the interval computed, the interval either contains μ or it does not — the probability is 0 or 1, we just do not know which. The randomness lives in the *sampling procedure*, not in μ.

(c) Correct interpretation: *"We are 95% confident that the mean daily sodium intake of all U.S. adults is between 3,210 mg and 3,480 mg."*

(d) The "95%" is a property of the **method**, not of this one interval: if the sampling and interval-construction procedure were repeated many times, 95% of the resulting intervals would capture the true μ. Out of 100 such samples, about **95** of the intervals would contain μ (and about 5 would miss it).

(e) The entire interval lies far above 2,300 mg, so every plausible value of μ exceeds the guideline. We can conclude that mean daily sodium intake is well above the recommended 2,300 mg — the shortfall is not a fluke of sampling.

Problem 4: Confidence Interval for a Mean, σ Unknown (15 points)

(a) $\sum x_i = 246.0$, so $\bar{x} = 246.0/12 = 20.50$ g. With $\sum(x_i - \bar{x})^2 = 2.62$,

$$s = \sqrt{\frac{2.62}{11}} = \sqrt{0.2382} = 0.488 \text{ g}, \qquad \frac{s}{\sqrt{n}} = \frac{0.488}{\sqrt{12}} = 0.141 \text{ g}.$$

(b) The population standard deviation σ is unknown and is being estimated by s from a small sample. That extra uncertainty is accounted for by using the t distribution, which has heavier tails than the standard normal. Degrees of freedom: $\text{df} = n - 1 = 11$.

(c) With $t^*_{0.025,\,11} = 2.201$:

$$20.50 \pm 2.201(0.141) = 20.50 \pm 0.310 \implies (20.190,\ 20.810) \text{ g}.$$

(d) With $t^*_{0.05,\,11} = 1.796$:

$$20.50 \pm 1.796(0.141) = 20.50 \pm 0.253 \implies (20.247,\ 20.753) \text{ g}.$$

The 90% interval is narrower (width 0.506 vs. 0.620) — less confidence buys more precision.

(e) The value 20 g lies **below** the entire 95% interval (20.190, 20.810). The data are not consistent with a mean of exactly 20 g; the bars appear to average slightly *more* protein than claimed. (This is good news for the consumer, and is the kind of margin a manufacturer often builds in deliberately.)

(f) The t interval requires the 12 observations to be a random sample from an approximately **normal** population (independence is assumed from the random sampling). With only $n = 12$, the CLT cannot be relied on, so normality matters. Check it by making a normal Q–Q plot of the 12 values and looking for points falling close to a straight line, supplemented by a histogram or dotplot and a check for outliers.

Problem 5: Determining the Sample Size for a Mean (10 points)

(a) With $z^* = 1.96$, $\sigma = 7$, $E = 2$:

$$n = \left(\frac{z^*\sigma}{E}\right)^2 = \left(\frac{1.96 \times 7}{2}\right)^2 = (6.86)^2 = 47.06 \implies n = 48.$$

(b) With $E = 1$:

$$n = \left(\frac{1.96 \times 7}{1}\right)^2 = (13.72)^2 = 188.24 \implies n = 189.$$

(c) $189/48 \approx 3.9$, i.e. about **4 times** as many students. Halving the margin of error requires quadrupling the sample size, because $E \propto 1/\sqrt{n}$ — the same $\sqrt{n}$ relationship that governs the standard error.

(d) Rounding down would give a sample size whose margin of error is *larger* than the requirement, so the stated precision would not be achieved. Rounding up guarantees the margin of error is at most E.

Problem 6: Confidence Intervals in R (8 points)

(a)

```
> mean(bar)
[1] 20.5
> sd(bar)
[1] 0.4880413
> t.test(bar)$conf.int
[1] 20.18993 20.81007
attr(,"conf.level")
[1] 0.95
```

This matches part 4(c) to three decimals.

(b) Add the `conf.level` argument:

```
> t.test(bar, conf.level = 0.90)$conf.int
[1] 20.24703 20.75297
```

(c) Any equivalent code is acceptable:

```
> xbar <- 14.8; sigma <- 1.2; n <- 36
> se <- sigma / sqrt(n)
> xbar + c(-1, 1) * qnorm(0.975) * se
[1] 14.40800 15.19200
```

Suggested Point Allocation (75 points)

Problem	Topic	Points	Breakdown
1	CLT conditions, independence	12	(a) 4, (b) 6, (c) 2
2	CI for a mean, σ known	15	(a) 2, (b) 4, (c) 4, (d) 3, (e) 2
3	Interpreting a CI	15	(a) 3, (b) 3, (c) 3, (d) 3, (e) 3

Problem	Topic	Points	Breakdown
4	CI for a mean, σ unknown	15	(a) 3, (b) 2, (c) 4, (d) 3, (e) 2, (f) 1
5	Sample size for a mean	10	(a) 3, (b) 3, (c) 2, (d) 2
6	R	8	(a) 3, (b) 2, (c) 3

Common errors to watch for. Problem 1(b1): counting 40 measurements as $n = 40$ — this is the exam's "verifying independence in practice" idea. Problem 2(e): accepting that known σ removes the shape condition. Problem 3(b): the probability-versus-confidence distinction is the single most-missed item; insist the answer says μ is fixed. Problem 4(b): using $z^* = 1.96$ instead of $t^* = 2.201$. Problem 5: rounding 47.06 down to 47.