
The Sampling Distribution of the Sample Mean and Confidence Intervals (Chap 5.2-5.3)

STAT 515 — Statistical Methods I

Dr. Yen-Yi Ho

Department of Statistics, University of South Carolina

1. The sampling distribution of the sample mean
2. The Central Limit Theorem
3. From the CLT to a confidence interval
4. Interpreting and adjusting confidence intervals

1. Two methods for statistical inference
 - Estimation
 - Hypothesis testing
2. Point estimation
3. Confidence interval estimation
4. Hypothesis testing
5. Review of estimation and hypothesis testing

■ Point estimation

- An **estimator** of a population parameter: a statistic (e.g. the sample mean, the sample proportion).
- An **estimate** of a population parameter: the value of the estimator for a particular sample.

■ Interval estimation

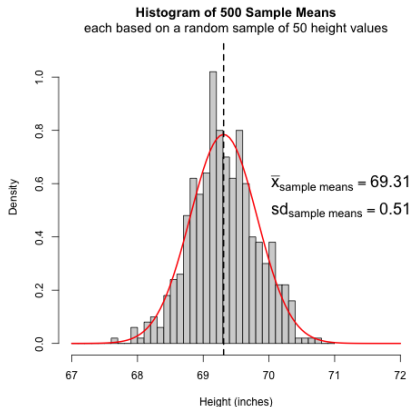
- A point estimate **plus** an interval that expresses the uncertainty or variability associated with the estimate.

Sampling Distribution of the Sample Mean

- In the previous section we reviewed the results of simulations that resulted in estimates of what's formally called the **sampling distribution** of a sample mean.
- The sampling distribution of a sample mean is a **theoretical probability distribution**; it describes the distribution of all sample means from all possible random samples of the same size taken from a population.

Sampling Distribution

For example: this histogram is an estimate of the sampling distribution of sample height means based on random samples of $n = 50$ from the population of all male students.



Semantic: Standard Deviation vs. Standard Error

- The term **standard deviation** refers to the variability in individual observations in a single sample (s) or population (σ).
- The **standard error of the mean** is also a measure of standard deviation, but not of individual values — rather variation in multiple **sample means** computed on multiple random samples of the same size, taken from the same population.

Sampling Distribution of the Sample Mean

- In real research it is **impossible** to estimate the sampling distribution of a sample mean by actually taking multiple random samples from the same population — no research would ever happen if a study needed to be repeated multiple times to understand this sampling behavior.
- Simulations are useful to illustrate a concept, but not to highlight a practical approach!
- Luckily, there is some mathematical machinery that generalizes some of the patterns we saw in the simulation results.

The Central Limit Theorem (CLT)

The **Central Limit Theorem (CLT)** is a powerful mathematical tool that gives several useful results:

- The sampling distribution of sample means based on all samples of the same size n is **approximately normal**, regardless of the distribution of the original (individual level) data in the population/samples.
- The mean of all sample means in the sampling distribution is the true mean of the population from which the samples were taken, μ .
- Standard deviation in the sample means of size n is equal to $\frac{\sigma}{\sqrt{n}}$: this is often called the **standard error** of the sample mean and sometimes written as $SE(\bar{x})$.

Example: Self-Reported Heights of Male Students

- Population distribution of individual height measurements for male students: normal
- True Population mean(μ) = 69.31 inches, sd(σ)=3.61 inches

	Means of 500 Sample Means	Means of 5000 Sample Means	SD of 500 Sample Means	SD of 5000 Sample Means	CLT
n=20	69.25	69.32	0.84	0.81	$\frac{3.61}{\sqrt{20}} \approx 0.81$
n=50	69.30	69.30	0.52	0.52	$\frac{3.61}{\sqrt{50}} \approx 0.51$
n=100	69.33	69.31	0.36	0.36	$\frac{3.61}{\sqrt{100}} \approx 0.36$

Example: Heights of Male Students

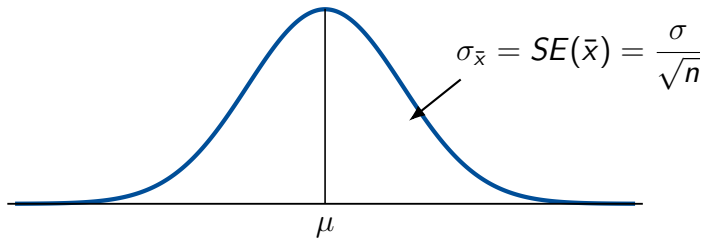
- Population distribution of individual BP measurements for males: normal
- True mean $\mu = 125$ mmHg; $\sigma = 14$ mmHg

Sample Sizes	Means of 500 Sample Means	Means of 5000 Sample Means	SD of 500 Sample Means	SD of 5000 Sample Means	SD of Sample Means (SE) by CLT
$n = 20$	124.98 mmHg	125.05 mmHg	3.31 mmHg	3.11 mmHg	3.13 mmHg
$n = 50$	125.03 mmHg	125.01 mmHg	1.89 mmHg	1.96 mmHg	1.98 mmHg
$n = 100$	124.99 mmHg	125.01 mmHg	1.43 mmHg	1.39 mmHg	1.40 mmHg

Recap: CLT

So the CLT tells us the following:

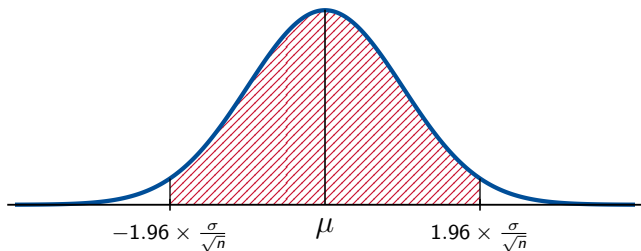
- When taking a random sample of continuous measures of size n from a population with true mean μ and true sd σ , the theoretical sampling distribution of sample means from all possible random samples of size n is as follows:



CLT: So What?

So what good is this info?

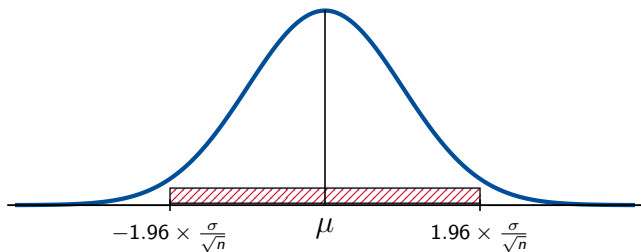
- Well, using the properties of the normal curve, this shows that for most random samples we can take (95%), the sample mean $\bar{x}$ will fall within 1.96 SEs of the true mean μ .



CLT: So What?

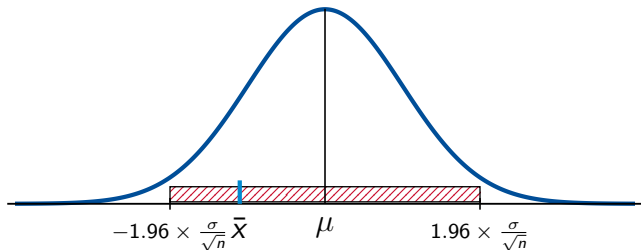
So AGAIN what good is this info?

- We are going to take a **single** sample of size n and get one $\bar{x}$.
- So we won't know μ — and if we did know μ , why would we care about the distribution of estimates of μ from imperfect subsets of the population?



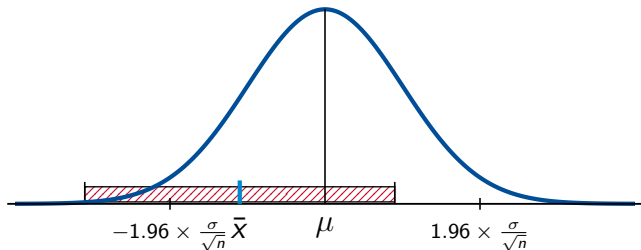
CLT: So What?

- We are going to take a single sample of size n and get one $\bar{x}$.
- But for most (95%) of the random samples we can get, our $\bar{x}$ will fall within ± 1.96 SEs of μ .



CLT: So What?

- We are going to take a single sample of size n and get one $\bar{x}$.
- So if we start at $\bar{x}$ and go 1.96 SEs in either direction, the interval created will contain μ most (95 out of 100) of the time.



Estimating a Confidence Interval

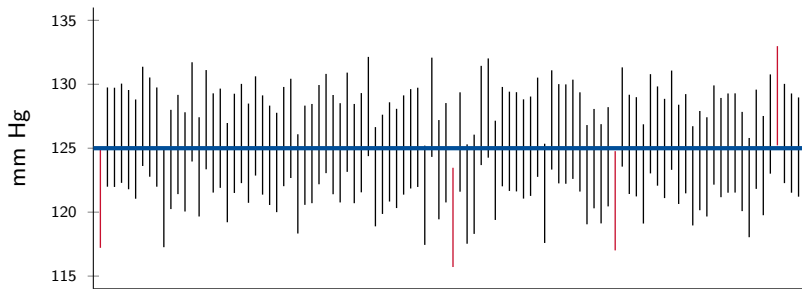
- Such an interval is called a **95% confidence interval** for the population mean μ .
- Interval given by

$$\bar{x} \pm 1.96 SE(\bar{x}) \longrightarrow \bar{x} \pm 1.96 \times \frac{\sigma}{\sqrt{n}}$$

- **Problem:** we don't know σ either.
 - Can estimate with s ; will detail this in the next section.
- What is the interpretation of a confidence interval?

Interpretation

One hundred 95% confidence intervals from 100 random samples of size $n = 50$:
Blood Pressure for Males.



Intervals drawn in **red** are the ones that miss the true mean $\mu = 125$ mmHg (blue line).

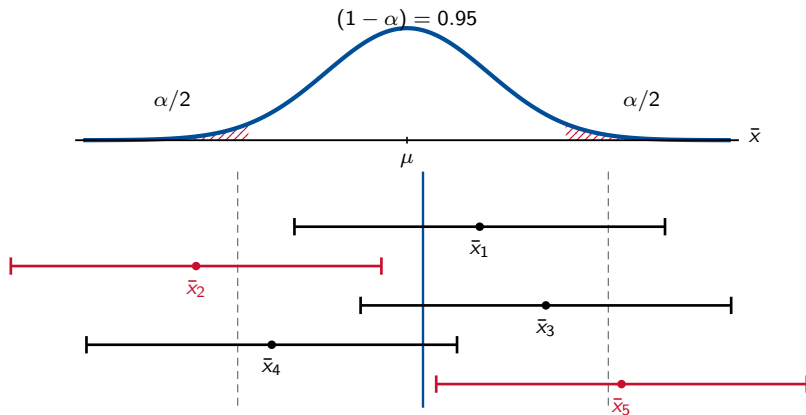
- **Fact:** when the data are an independent sample from a normally distributed population with mean μ and **known** variance σ^2 , the interval

$$\bar{x} \pm z_{\alpha/2} \cdot \sigma_{\bar{x}}$$

will contain the true population mean μ with probability 0.95 (taking $z_{\alpha/2} = 1.96$).

- By the **Central Limit Theorem**, this is approximately true when the data are independently sampled from *any* distribution with finite variance σ^2 when n is large enough.

Graphical Illustration for $100(1 - \alpha)\%$ CI



Intervals drawn in **red** fail to cover μ .

Interpretation of a 95% Confidence Interval

1. In **repeated sampling** from a normally distributed population, $100(1 - \alpha)\%$ of all intervals of this form will, in the long run, include the population mean μ .
2. The probability that the interval covers the population mean μ is $100(1 - \alpha)\%$:
 - “We are $100(1 - \alpha)\%$ confident that the interval covers the population mean μ .”

Careful

The **interval** is random, not μ . Once the data are observed, a particular interval either covers μ or it does not.

■ Random sampling error

- A confidence interval only accounts for random sampling error — not other systematic sources of error or bias.

■ Examples of Systematic Bias

- BP measurement is always +5 too high (broken instrument).
- Only those with high BP agree to participate (non-response bias).

- Are all CIs 95%?
 - No
 - It is the most commonly used
 - A 99% CI is

- Are all CIs 95%?
 - No
 - It is the most commonly used
 - A 99% CI is **wider**
 - A 90% CI is

- Are all CIs 95%?
 - No
 - It is the most commonly used
 - A 99% CI is **wider**
 - A 90% CI is **narrower**
- To change the level of confidence, adjust the number of SEs added to and subtracted from $\bar{x}$:
 - For a 99% CI, you need ± 2.58 SE
 - For a 95% CI, you need ± 1.96 SE
 - For a 90% CI, you need ± 1.65 SE

Exercise 1: CI for μ (Known σ)

College Class-to-Class Time

A study records the time between classes for $n = 80$ students. It is known that $\sigma = 1.5$ min. The sample mean is $\bar{x} = 3.4$ min. Construct a 95% CI for the true mean time.

Solution: σ is known and $n = 80 \geq 30 \Rightarrow$ use $z_{0.025} = 1.96$.

$$\bar{x} \pm z_{0.025} \cdot \frac{\sigma}{\sqrt{n}} = 3.4 \pm 1.96 \times \frac{1.5}{\sqrt{80}} = 3.4 \pm 0.329 = (3.071, 3.729) \text{ min}$$

Interpretation

We are **95% confident** that the true mean time between classes is between 3.07 and 3.73 minutes.

Determining Needed Sample Size — Mean

Key idea: the **half-width** (sampling error, SE) of a CI is

$$SE = z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}$$

Solving for n :

Required Sample Size for μ

$$n = \left(\frac{z_{\alpha/2} \sigma}{SE} \right)^2$$

If σ is unknown, substitute s from a pilot study. Always **round up** to the next integer.

Example

Want a 95% CI with $SE = 1$ and $\sigma \approx 15$.

$$n = \left(\frac{1.96 \times 15}{1} \right)^2 = (29.4)^2 = 864.36 \Rightarrow n = 865$$

Exercise 6: Sample Size for NFL Footballs

NFL Football Pressure

Want a 99% CI for mean pressure with $SE = 0.025$ psi. Prior study: $\sigma \approx 5$ psi.

Solution: $1 - \alpha = 0.99$, so $z_{0.005} = 2.576$.

$$n = \left(\frac{2.576 \times 5}{0.025} \right)^2 = (515.2)^2 = 265,431 \Rightarrow n = 265,432$$

That's a large sample!

A very tight SE (0.025 psi) with high confidence (99%) requires a very large n . There is always a trade-off between precision, confidence, and cost.