Chapter 4

Lecture 4: Probability

Yen-Yi Ho

Department of Statistics
University of South Carolina

Class 4 Outline

1. Basic Probability Concepts
2. Rules of Probability
3. Other Rules of Probability
4. Examples Using Probability
5. Bayes' Rule

Basic Probability Concepts

Probability provides a measure of the **uncertainty** associated with the occurrence of events.

1. Theoretical probability
2. Long-run relative frequency
3. Subjective probability

Theoretical Probability

$$\Pr(E) = \frac{m}{N}$$

If an event can occur in N mutually exclusive and equally likely ways, and m of these possess characteristic E, then $\Pr(E) = m/N$.

Example: Flip 2 coins

$N = 4$ equally likely outcomes: HH, HT, TH, TT. For two heads, $m = 1$, so

$$\Pr(\text{two heads}) = \frac{1}{4} = .25.$$

Long-run Relative Frequency

Long-run Relative Frequency

$$\frac{m}{n} \simeq \Pr(E) \quad \text{for large } n.$$

If a process is repeated many times, the relative frequency of E approaches its probability.

Example: Flip 2 coins 100 times

HH: 22, HT: 28, TH: 23, TT: 27. Thus

$$\Pr(\text{two heads}) \simeq 22/100 = .22 \simeq .25.$$

Long-run Relative Frequency Example

Flip 2 coins 10,000 times:

Outcome	Frequency
HH	2,410
HT	2,582
TH	2,404
TT	2,604
Total	10,000

$$\Pr(\text{two heads}) \simeq 2410/10000 = .241 \simeq .25.$$

The probability is defined as the limiting long-run relative frequency as $n \to \infty$.

Subjective Probability

- Probability ranges from 0 to 1.
- Subjective probability expresses uncertainty when equally likely outcomes or long-run repetitions are not directly available.

`https://www.youtube.com/watch?v=-3QldN3EnBI&list=WL&index=6`

Random Experiment

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Example:

A fair coin is flipped twice.

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Sample Space and Events

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The set S of all the possible outcomes (sample points) ω of the experiment. Sample spaces can be either discrete or continuous.

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Discrete sample space
S:{HH, HT, TH, TT}

Continuous sample space
S:{[0, 60]}

Definition: Event

An **event** (usually denoted by capital letters at the alphabet) is a specific collection of sample points.

A=observing at least one head after flipping a fair coin twice
= HH, HT, TH

B= Taking at least 30 minutes to finish an one-hour exam.
$= \{\omega : \omega \in [30, 60]\}$

Assigning Probabilities: The Coin-Flipping Example

The coin flipping example is discrete, so we can define the probability measure by giving a probability to each of the sample points so that the sum is 1.

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$$P(S) = 1.$$

$$S : \{HH, HT, TH, TT\}$$

what is $P(HH) = ?$

Probability of an Event

We would get the probability of an event simply by adding the probabilities assigned to their sample points.

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A$=$ Exactly one head $= \{$(HT), (TH) $\}$
$P(A) = P(HT) + P(TH) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

B$=$First flip was a head $=$?

Exercise 1

Example: Coin and Die

Consider the experiment of flipping a fair coin once and rolling a fair six-sided die once. Let the event $A = \{\text{a head and an odd number occurred}\}$.

a.) What is the sample space for this experiment?

b.) What is the probability associated with each sample point?

c.) What sample points make up the event A?

d.) $P(A) = ?$

Equiprobability Model

Example: Equiprobability Model

The **equiprobability model** (with all outcomes in S equally likely) does not always hold.

Example: Chlamydia Testing

A USC undergraduate student is tested for chlamydia ($0 =$ negative, $1 =$ positive):

$$S = \{0, 1\}.$$

It is not reasonable to assume that each outcome in S is equally likely, because the prevalence of chlamydia among college age students is much less than 50 percent (in SC, this prevalence is probably somewhere between 1–5 percent).

Null Event

Definition: Null Event

The **null event**, denoted by $\emptyset$, is an event that contains no outcomes (therefore, the null event cannot occur). The null event has probability $P(\emptyset) = 0$.

Example:

For example, tossing a fair die, it's impossible to observe a number that is both odd and even.

Union of Two Events

Definition: Union

The **union** of two events A and B contains all outcomes ω in either event or in both. We denote the union of two events A and B by

$$A \cup B = \{\omega : \omega \in A \text{ or } \omega \in B\}.$$

Example: Rolling a Fair Die

$A =$ observing an even number.
$B =$ observing a number greater than 3.
$A \cup B =$ observing an even number or a number greater than 3.

Intersection of Two Events

Definition: Intersection

The **intersection** of two events A and B contains all outcomes ω in both events. We denote the intersection of two events A and B by

$$A \cap B = \{\omega : \omega \in A \text{ and } \omega \in B\}.$$

Example: Rolling a Fair Die

$A =$ observing an even number.
$B =$ observing a number greater than 3.
$A \cap B =$ observing an even number that is greater than 3.

Mutually Exclusive Events

Definition: Mutually Exclusive (Disjoint) Events

If the events A and B contain no common outcomes, we say the events are **mutually exclusive** (or disjoint). In this case,

$$P(A \cap B) = P(\emptyset) = 0.$$

Rules of Probability

For mutually exclusive outcomes $E_1, \ldots, E_n$:

1. $\Pr(E_i) \geq 0$.
2. $\sum_i \Pr(E_i) = 1$.
3. If E_i and E_j are mutually exclusive,

$$\Pr(E_i \text{ or } E_j) = \Pr(E_i) + \Pr(E_j).$$

Other Rules of Probability

Addition Rule

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B).$$

Conditional Probability

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)}, \qquad \Pr(B) \neq 0.$$

Multiplication Rule

$$\Pr(A \cap B) = \Pr(B)\Pr(A \mid B).$$

If A and B are **independent**, $\Pr(A \mid B) = \Pr(A)$ and $\Pr(A \cap B) = \Pr(A)\Pr(B)$.

Addition Rule

Addition Rule

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Best demonstrated using Venn diagrams.

Example Using Probability

- $\Pr(\text{male}) = 50/100 = .5$
- $\Pr(\text{female}) = 50/100 = .5$
- $\Pr(\text{young}) = 70/100 = .7$
- $\Pr(\text{old}) = 30/100 = .3$

	Young	Old	Total
Male	30	20	50
Female	40	10	50
Total	70	30	100

Example Using Probability (cont'd)

Using the same 2×2 table:

- $\Pr(\text{female and old}) = 10/100 = .1.$
- By the addition rule,

$$\begin{aligned}\Pr(\text{male or young}) &= \Pr(\text{male}) + \Pr(\text{young}) - \Pr(\text{male and young}) \\ &= .50 + .70 - .30 = .90.\end{aligned}$$

Conditional Probability

Definition: Conditional Probability

For two events A and B with $P(B) > 0$, the **conditional probability** of A, given that B has occurred, is

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}.$$

Independence

Definition: Independence

Two events A and B are **independent** if any of the following equivalent statements is true:

1. $P(A \mid B) = P(A)$
2. $P(B \mid A) = P(B)$
3. $P(A \cap B) = P(A)P(B)$

Example Using Probability

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	Young	Old	Total
Male	30	20	50
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- $\Pr(\text{old} \mid \text{female}) = 10/50 = .2$.
- $\Pr(\text{old} \mid \text{male}) = 20/50 = .4$.
- $\Pr(\text{old}) = 30/100 = .3$.

Therefore sex and age are **not independent** in this group because

$$\Pr(\text{old} \mid \text{female}) \neq \Pr(\text{old} \mid \text{male}) \neq \Pr(\text{old}).$$

Example 2

Example: Flip a Fair Coin Twice

A = Exactly one Head = $\{(HT), (TH)\}$
B = First flip was a Head = $\{(HH), (HT)\}$
$A \cap B$ = Exactly one head observed and the first flip was a head = $\{HT\}$

$P(A) = 0.5, \quad P(B) = 0.5, \quad P(A \cap B) = 0.25$

Then, the conditional probability of observing exactly one head given that the first flip was a head is,

$$P(A \mid B) = \ ?$$

Example 3

Example: High Blood Pressure

For undergraduate students, the probability of having high blood pressure is .03. For male undergraduates, the probability of having high blood pressure is .04. For undergraduate students, are "being male" and "having high blood pressure" independent? Show why or why not?

Rule of Complements

Rule of Complements

$$P(A) + P(A^c) = 1 \implies P(A^c) = 1 - P(A)$$

The complement of an event A is the event that A does not occur — that is, the event consisting of all sample points that are not in event A. We denote the complement of A by A^c.

Mutual Independence

Definition: Mutual Independence

The events $A_1, \ldots, A_n$ are **mutually independent** if and only if for every subcollection of these events

$$P(A_{i1} \cap A_{i2} \cap \cdots \cap A_{ik}) = P(A_{i1}) \cdot P(A_{i2}) \cdots P(A_{ik}).$$

Multiplication Rule

$$P(A \cap B) = P(A \mid B) \cdot P(B) \quad \text{or} \quad = P(B \mid A) \cdot P(A)$$

Best thought about using tree diagrams.

Exercise 4

Example: Basketball Free Throws

Consider a basketball player attempting two free throws, with a 60% chance of making each one, and the two throws are independent.

(a) What's the probability that he makes the two throws?

(b) What's the probability that he makes the second throw?

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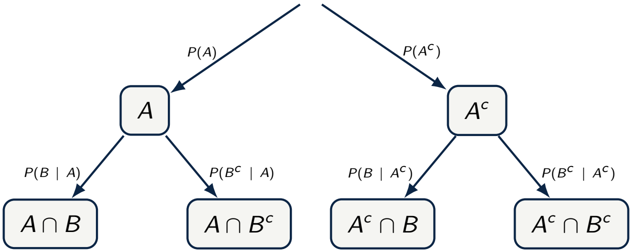

Exercise 3: Basketball Free Throws (Independent)

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$M_1 = \{\text{making the first throw}\}, \quad M_2 = \{\text{making the second throw}\}$

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(a) $P(M_1 \cap M_2) = P(M_1) \cdot P(M_2) = .6(.6) = .36$

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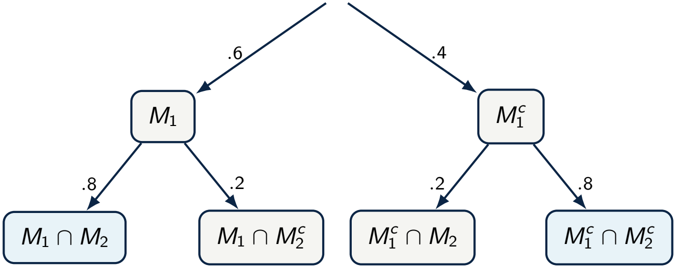
(b) $P(M_2) = .6$

What if the Two Throws Were Dependent?

Example:

Say the chance of making the first was 60%, but that the chance of making the second is 80% if he made the 1^{st} and only 20% if he missed.

$$P(M_2 \mid M_1) = 80\% = .8 \qquad P(M_2 \mid M_1^c) = 20\% = .2$$



Dependent Throws: Solutions

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a) $P(M_1 \cap M_2) = P(M_1) \cdot P(M_2 \mid M_1) = .6(.8) = .48$

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b)

$$\begin{aligned} P(M_2) &= P(M_1 \cap M_2) + P(M_1^c \cap M_2) \\ &= .48 + P(M_1^c) \cdot P(M_2 \mid M_1^c) \\ &= .48 + .4(.2) = .48 + .08 \\ &= .56 \end{aligned}$$

Total Probability Rule

Total Probability Rule

Suppose that A and B are two events, then

$$P(A) = P(A \cap B) + P(A \cap B^c)$$

$$= P(A \mid B)P(B) + P(A \mid B^c)P(B^c)$$

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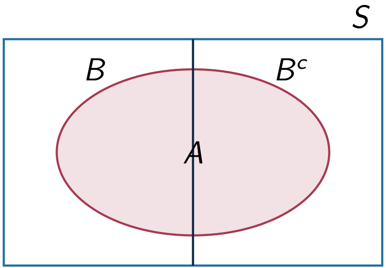

$B \cup B^c = S$

Total Probability Rule (General Version)

Definition: Total Probability Rule (general version)

Let $B_1, B_2, \ldots, B_n$ be disjoint and exhaustive events (that is, $B_i \cap B_j = \emptyset$ for $i \neq j$, $\bigcup_{i=1 \text{ to } n} B_i = S$).

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Then for any A,

$$\begin{aligned} P(A) &= P(A \cap B_1) + P(A \cap B_2) + \cdots + P(A \cap B_n) \\ &= P(A \mid B_1)P(B_1) + P(A \mid B_2)P(B_2) + \cdots + P(A \mid B_n)P(B_n) \end{aligned}$$

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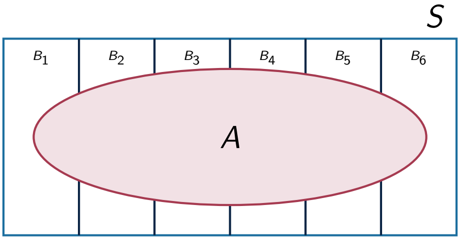

Exercise 4: Defective Products

Example: Machines I, II, and III

Machines I, II, and III produce the same product. $D = \{\text{being defective}\}$. The rates of defectives from each are 2%, 1%, and 3% respectively.

$$P(D \mid I) = 2\%, \quad P(D \mid II) = 1\%, \quad P(D \mid III) = 3\%$$

The percent of the total product made by each are 35%, 25%, and 40% respectively.

$$P(I) = .35, \quad P(II) = .25, \quad P(III) = .4$$

What percent of the product are defective?

Exercise 4: Solution

By the total probability rule (multiplicative rule),

$$\begin{aligned} P(D) &= P(D \cap I) + P(D \cap II) + P(D \cap III) \\ &= P(I) \cdot P(D \mid I) + P(II) \cdot P(D \mid II) + P(III) \cdot P(D \mid III) \end{aligned}$$

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$$= .35(.02) + .25(.01) + .4(.03)$$

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$$= .35(.02) + .25(.01) + .4(.03)$$

$$= 0.0215 = 2.15\%$$

Class 4 Outline

Bayes' Rule

- Useful for computing $\Pr(B \mid A)$ when $\Pr(A \mid B)$ is known.
- Screening example: sensitivity gives $\Pr(\text{test}+ \mid \text{disease})$, while practice often asks for $\Pr(\text{disease} \mid \text{test}+)$.

Bayes' Rule

$$\Pr(B \mid A) = \frac{\Pr(A \mid B)\Pr(B)}{\Pr(A)} = \frac{\Pr(A \mid B)\Pr(B)}{\Pr(A \mid B)\Pr(B) + \Pr(A \mid \bar{B})\Pr(\bar{B})}.$$

Bayes' Example

Suppose a disease test has known sensitivity and specificity, and disease prevalence is known. Compute the **positive predictive value**. Given

$$\Pr(D) = .2, \quad \Pr(\bar{D}) = .8,$$
$$\Pr(T \mid D) = .90, \quad \Pr(\bar{T} \mid D) = .10,$$
$$\Pr(\bar{T} \mid \bar{D}) = .85, \quad \Pr(T \mid \bar{D}) = .15.$$

We want $\Pr(D \mid T)$.

Bayes' Example: Complete the 2×2 Table

For a hypothetical population of 100:

	D	$\bar{D}$	Total
Test $+$	18	12	30
Test -	2	68	70
Total	20	80	100

The counts follow directly from prevalence, sensitivity, and specificity.

Bayes' Example: Positive Predictive Value

From the completed table,

$$\Pr(D \mid T) = \frac{18}{30} = 0.60.$$

Thus, among those who test positive, 60% have the disease in this example.

Bayes' Example Using the Formula

$$
\begin{aligned}
\Pr(D \mid T) &= \frac{\Pr(T \mid D)\Pr(D)}{\Pr(T \mid D)\Pr(D) + \Pr(T \mid \bar{D})\Pr(\bar{D})} \\
&= \frac{(.90)(.20)}{(.90)(.20) + (.15)(.80)} \\
&= 0.60.
\end{aligned}
$$