Chapter 4

Lecture 5: Counting Techniques

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Lecture 5 Outline

- Factorial
- Permutation
- Partition
- Combination

Motivation to Study Counting Rules

Motivation to study counting rules

If all outcomes in S are equally likely, for any event $A \subset S$, we have

$$P(A) = \frac{\#\{\text{sample points in } A\}}{\#\{\text{sample points in } S\}}.$$

This leads to an interest in **counting rules**.

To use this formula, we need to be able to *count* sample points — without necessarily listing them all.

Fundamental Theorem of Counting

Definition: Fundamental theorem of counting

If a job consists of K tasks such that the tasks may be completed in $n_1, \ldots, n_K$ ways, respectively, then there are

$$\prod_{k=1}^{K} n_k = n_1 \times n_2 \times \cdots \times n_K$$

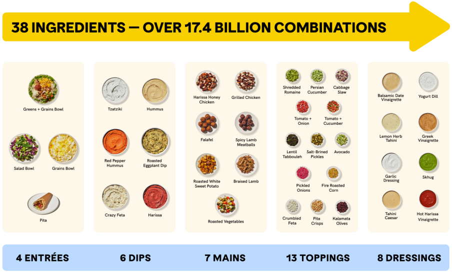
ways to do the job.

At CAVA...

Example: Setting Up the Cava Choices

1. Base (2 options): Rice, Greens
2. Protein (2 options): Chicken, Falafel
3. Dressing (3 options): Garlic Dressing, Tahini, Tzatziki

In how many ways can you build your meal?



Factorials

- Notation: "n factorial" = n! = number of possible arrangements of n objects.
- $n! = n(n\text{-}1)(n\text{-}2)(n\text{-}3) \ldots (2)(1)$.
- $0! = 1$.
- Example: $4! = 4 \times 3 \times 2 \times 1 = 24$ arrangements.

Permutations

Definition: Permutation

Draw r elements of N elements *without replacement* and *arrange them in some order*. The $\#$ ways is

$$N(N-1)\cdots(N-r+1) = \frac{N!}{(N-r)!}.$$

Example: Voting at a club

Choosing a President, Vice President, and Treasurer from a 10-person club, in how many ways can these roles be assigned?

Partition

Definition: Partition

Place N balls into K urns such that the K urns receive $n_1, \ldots, n_K$ balls, where $N = n_1 + \cdots + n_K$. The # ways is

$$\frac{N!}{n_1!\, n_2! \cdots n_K!}.$$

Example: Assigning passengers to vehicles

Suppose 12 people are randomly assigned to ride in 3 vehicles taking 4, 5, and 3 passengers, respectively.

1. In how many ways can the passengers be assigned to the different vehicles?
2. If you and your friend are among these people, with what probability will the two of you ride in the same vehicle?

Combination

Definition: Combination

Draw r elements from N *without replacement* and *without regard to their order*. The $\#$ ways is

$$\binom{N}{r} = \frac{N!}{r!(N-r)!}.$$

[note: the quantity $0! = 1$]

This is just a partition with only 2 "urns."

Example:

In how many ways could you choose 2 of the 6 novels to read?

Combinations Rule

Example:

How many different ways we can choose 2 students from 8 students? $N = 8,\ n = 2$

$$\binom{8}{2} = \frac{8!}{2!\,6!} = \frac{8 \times 7}{2 \times 1} = 28$$

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$$\binom{8}{1} = \frac{8!}{1!\,7!} = 8, \qquad \binom{8}{0} = \frac{8!}{0!\,8!} = 1$$

Exercise: Card Hands

Example:

If dealt 5 cards from a 52-card deck, what is the probability of getting

1. the ace of diamonds?
2. at least one ace?

Exercise 5: Component Defects

Example: Components Shipped in Lots

Components are known to have a defective rate of **0.02** (2%) and are shipped in lots of 20. Assume the 20 components are independent.

$$D_1 = \{\text{component 1 being defective}\}, \quad D_2 = \{\text{component 2 being defective}\}, \ldots$$

$$P(D_1) = .02, \quad P(D_1^c) = .98$$

What is the probability that the entire lot of 20 contains no defectives?

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$$P(D_1^c \cap D_2^c \cap \cdots \cap D_{20}^c) \overset{\text{by independence}}{=} P(D_1^c) \cdot P(D_2^c) \cdots P(D_{20}^c)$$
$$= .98^{20} = 1.5546 \times 10^{-12} \approx 0$$

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$$P(D_1 \cap D_2^c \cap D_3^c \cap \cdots \cap D_{20}^c) = .02 \cdot (.98)^{19}$$

There are $\binom{20}{1} = 20$ ways to choose which component is the defective one, so

$$\begin{aligned} P(D_1^c \cap D_2 \cap D_3^c \cap \cdots \cap D_{20}^c) &= P(D_1^c) \cdot P(D_2) \cdot P(D_3^c) \cdots P(D_{20}^c) \\ &= .98 \cdot .02 \cdot .98^{19} \end{aligned}$$

$$P(\text{exactly one defective}) = 20 \cdot (.02)(.98)^{19} = .2725$$

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Counting the Ways: Exactly 10 out of 20

In order to determine the probability of having exactly 10 out of 20 defectives, we would need to have some way of easily counting the number of ways this can happen.

Example:

```
YYYYYYYYYYNNNNNNNNNN
YNYYYYYYYYYNNNNNNNNN
YYNYYYYYYYYNNNNNNNNN
```

etc. . .

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`YYYYYYYYYYNNNNNNNNNN`
`YNYYYYYYYYNNNNNNNNNN`
`YYNYYYYYYYNNNNNNNNNN`
etc.. . .

$$P(D_1 \cap D_2 \cap \cdots \cap D_{10} \cap D_{11}^c \cap D_{12}^c \cap \cdots \cap D_{20}^c) = .02^{10}(.98)^{10}$$