Lecture 6

The Poisson Random Variable & Continuous Random Variables

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The Poisson Random Variable

A type of probability distribution that is often useful in describing the number of **rare events** that will occur during a specific period or in a specific area or volume is the **Poisson distribution**.

Examples of a Poisson Random Variable

Example:

Examples of random variables for which the Poisson probability distribution provides a good model are

- The number of traffic accidents per month at a busy intersection
- The number of noticeable surface defects (scratches, dents, etc.) found by quality inspectors on a new automobile
- The number of death claims received per day by an insurance company

Characteristics of a Poisson Random Variable

- The experiment consists of counting the number of times a certain event occurs during a given unit of time or in a given area or volume (or weight, distance, or any other unit of measurement).
- The probability that an event occurs in a given unit of time, area, or volume is the same for all the units.
- The number of events that occur in one unit of time, area, or volume is independent of the number of that occur in other units.
- The mean (or expected) number of events in each unit is denoted by the Greek letter λ.

Poisson Random Variable

X = the # of rare events occur in a unit of time, area, or volume.

Poisson Probability Mass Function

$$P(X = x) = p(x) = \frac{\lambda^x e^{-\lambda}}{x!}, \quad \text{for } x = 0, 1, \ldots$$

$$E(X) = \mu_X = \lambda \qquad \mathrm{Var}(X) = \sigma_X^2 = \lambda$$

$$X \sim \mathrm{Poisson}(\lambda)$$

Example 6: Airline Fatalities

Example:

U.S. airlines average about 1.6 fatalities per month. (*Statistical Abstract of the United States: 2010*). Assume that the probability distribution for X, the number of fatalities per month, can be approximated by a Poisson probability distribution.

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$$X \sim \text{Poisson}(\lambda = 1.6)$$

a) What is the probability that no fatalities will occur during any given month?

$$P(X = 0) = p(0) = \frac{1.6^0 e^{-1.6}}{0!} = 1 \cdot e^{-1.6} = .202$$

Example 6 (cont'd)

b) What is the probability that at least two fatalities will occur during any given month?

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$$\mu_X = E(X) = 1.6 \qquad \sigma_X = \sqrt{\text{Var}(X)} = \sqrt{\sigma_X^2} = \sqrt{1.6} = 1.265$$

$$\mu_X \pm 2\sigma_X = 1.6 \pm 2(1.265) = 1.6 \pm 2.53 \approx (0, 4.13)$$

The probability of between 0 and 2 fatalities is at least $1 - \frac{1}{2^2} = .75$.

Example 7: Trees Struck by Lightning

Example:

The number of trees struck by lightning in one portion of a forest preserve approximately follows a Poisson process with a mean of 0.2 per acre. Assume we had some effective way of identifying the struck trees from the available imaging capabilities and that 6 acres are examined.

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$$X = \text{the \# of trees struck in 6 acres} \sim \text{Poisson}(\lambda = 1.2)$$

a) Find the expected number of trees that would be struck in the 6 acres that are examined.

$$E(X) = 6(.2) = 1.2$$

Example 7 (cont'd)

b) Find the probability that exactly 1 struck tree will be found in the 6 acres examined.

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$$P(X = 1) = p(1) = \frac{1.2^1 e^{-1.2}}{1!} = .361$$

Definition: Continuous Random Variable

A **continuous random variable** assumes values corresponding to **any of points** contained in an or more intervals.

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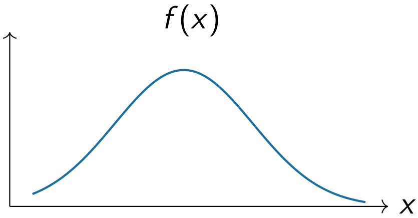
For a continuous random variable X:

$$P(X = x) = 0, \quad \text{for any } x$$

5.1 Continuous Probability Distributions

- The Graphical form of the probability distribution for a continuous random variable X is a smooth curve.



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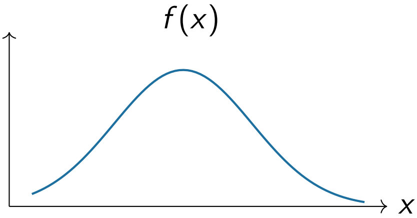
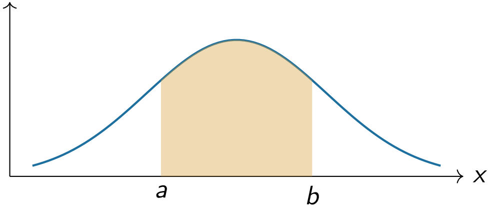


- This curve, a function of x, is denoted by $f(x)$ and is called a **probability density function**, or a probability distribution.

Areas Under the Curve

The areas under the probability distribution (the curve) correspond to probabilities for X.



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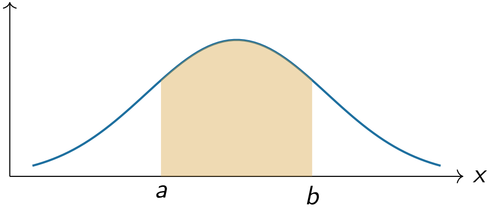

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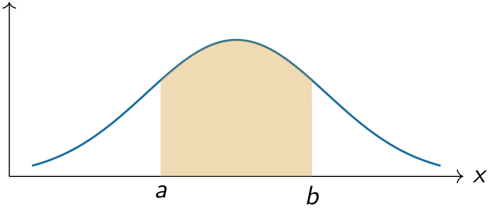

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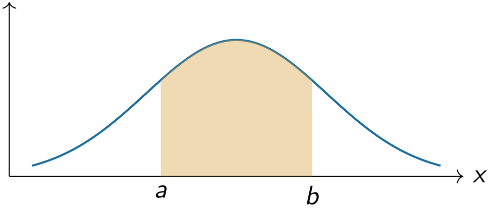


- All area under the curve $= 1$

$$P(a < X < b) = \int_a^b f(x)\,dx$$

$$P(X = a) = 0$$

$$P(X \leq b) = P(-\infty < X \leq b) = \int_{-\infty}^b f(x)\,dx = P(X < b)$$

A Probability Density Function (pdf)

Definition: pdf

$f(x)$ satisfies the following:

a) $f(x) \geq 0$ for all x.

b) $\displaystyle\int_{-\infty}^{+\infty} f(x)\,dx = 1$ (whole area under the density curve equal to 1)

c) $\displaystyle P(a < X < b) = \int_a^b f(x)\,dx$

Mean and Variance of a Continuous Random Variable

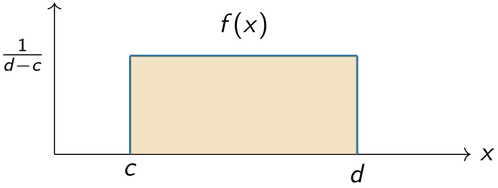
For mean and variance, we need to change the summation into an integral:

$$\mu = \sum x\, p(x) \quad \Longrightarrow \quad \mu = \int_{-\infty}^{+\infty} x f(x)\, dx$$

$$\sigma^2 = \sum (x-\mu)^2 p(x) \quad \Longrightarrow \quad \sigma^2 = \int_{-\infty}^{+\infty} (x-\mu)^2 f(x)\, dx$$

5.2 The Uniform Distribution

- The uniform probability distribution provides a model for continuous random variables that are evenly distributed over a certain interval.



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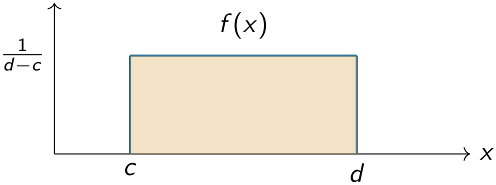

- There is no clustering of values around any one value; instead, there is an even spread over the entire region of possible values.

For a Uniform Random Variable X

$$X \sim \text{Uniform}(c, d)$$

Probability Density Function

$$f(x) = \frac{1}{d - c}, \quad c \leq x \leq d.$$

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- Mean: $\mu = \dfrac{c+d}{2}$
- standard deviation: $\sigma = \dfrac{d-c}{\sqrt{12}}$
- $P(a < X < b) = \dfrac{b-a}{d-c}, \quad c \le a < b \le d$

Example: Used-Car Warranty

Example:

An unprincipled used-car dealer sells a car to an unsuspecting buyer, even though the dealer knows that the car will have a major breakdown within the next 6 months. The dealer provides a warranty of 45 days on all cars sold. Let X represent the length of time until the breakdown occurs. Assume that X is a uniform random variable with values between 0 and 6 months.

$$X \sim \text{Uniform}(0, 6), \qquad f(x) = \frac{1}{6}, \quad 0 \leq x \leq 6$$

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a) Calculate the mean and standard deviation of X.

$$\mu_X = \frac{0+6}{2} = 3, \qquad \sigma_X = \frac{6-0}{\sqrt{12}} = 1.73$$

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a) Calculate the mean and standard deviation of X.

$$\mu_X = \frac{0+6}{2} = 3, \qquad \sigma_X = \frac{6-0}{\sqrt{12}} = 1.73$$

b) Calculate the probability that the breakdown occurs while the car is still under warranty (45 days $\approx$ 1.5 months).

$$P(X \leq 1.5) = \frac{1.5-0}{6-0} = \frac{1.5}{6} = .25$$