Lecture 9

The Exponential Distribution

Yen-Yi Ho

Department of Statistics
University of South Carolina

Outline

- Continuous Probability Distributions
  - The Exponential Distribution
  - Relationship Between Poisson and Exponential Distributions

The Exponential Distribution

The length of time or the distance between occurrences of random events can often be described by the **exponential probability distribution**.

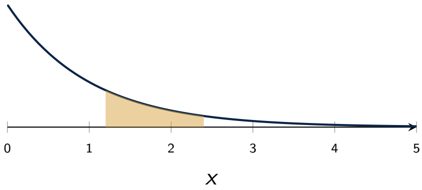
- the length of time between emergency arrivals at a hospital,
- the length of time between breakdowns of manufacturing equipment,
- the length of time between catastrophic events (floods, earthquakes, etc.).

For an Exponential Random Variable X

Probability Density Function

$$f(x) = \frac{1}{\theta} e^{-x/\theta}, \qquad x > 0.$$

- Mean: $\mu = \theta$
- Standard deviation: $\sigma = \theta$
- $P(X \geq a) = e^{-a/\theta}$, for $a > 0$



Example: Microwave Oven Warranty

A manufacturer of microwave ovens is trying to determine the length of warranty period it should attach to its magnetron tube, the most critical component in the oven. Preliminary testing has shown that the length of life (in years), X, of a magnetron tube has an exponential probability distribution with $\theta = 6.25$.

Example: (a) Find the mean and standard deviation of X

$$X \sim \text{Exponential}(\theta = 6.25)$$

$$\mu_X = E(X) = \theta = 6.25 \text{ (years)}, \qquad \sigma_X = \theta = 6.25 \text{ (years)}$$

Example: Microwave Oven Warranty (cont'd)

Example: (b) Warranty period of five years

Suppose a warranty period of five years is attached to the magnetron tube. What fraction of tubes must the manufacturer plan to replace, assuming the exponential model with $\theta = 6.25$ is correct?

$$P(X \leq 5) = 1 - e^{-5/6.25} = 1 - e^{-0.8} = 1 - 0.449 = 0.551$$

Example: (c) Replacing only 27.5% of tubes

What should the warranty period be if the manufacturer wants to replace only 27.5% of tubes?

$$P(X \leq a) = 0.275 \;\Rightarrow\; 1 - e^{-a/6.25} = 0.275 \;\Rightarrow\; e^{-a/6.25} = 0.725$$

$$a = 2.01 \text{ years} \qquad \text{(i.e., the 27.5th percentile)}$$

Relationship Between Poisson and Exponential Distributions

- If the number of occurrences per unit time follows a **Poisson distribution** with parameter λ, then the time between two successive occurrences (or the time until the first occurrence) Y follows an **Exponential distribution** with parameter

$$\theta = \frac{1}{\lambda}.$$

Example: Product Help Line

The number of customers calling a product help line approximately follows a Poisson process with mean of 8 calls per hour between 2am and 5am. $X \sim \text{Poisson}(\lambda = 8)$.

Example: (a) Expected time until the first call

Y = the amount of time until the first call should arrive.

$$Y \sim \text{Exponential}\left(\theta = \frac{1}{\lambda} = \frac{1}{8}\right), \qquad E(Y) = \theta = \frac{1}{8} \text{ (hour)}$$

Example: (b) Probability the first call arrives within 6 minutes

6 minutes is $1/10$th of an hour.

$$P\left(Y \le \tfrac{1}{10}\right) = 1 - e^{-\frac{1/10}{1/8}} = 1 - e^{-0.8} = 0.551$$