STAT 206: Chapter 8 (Confidence Interval Estimation)

Ideas in Chapter 8:

- Construct and interpret confidence interval estimates for:
  - Population proportion, p (or π if you prefer), using $\hat{p}$ and standard deviation of the $\hat{p}$ values, (standard error), $\sqrt{\frac{p(1-p)}{n}} \approx \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$
  - Population mean, μ, using the point estimate, $\bar{X}$ and the standard error of the mean, $\sigma_{\bar{X}} \approx s_{\bar{X}}$
- To determine the sample size necessary to develop a confidence interval for the population mean or population proportion
- How confident do we need to be? And what are the effects of higher confidence?

- **Confidence Intervals – General discussion**
  - Point estimate
    - Statistic used to estimate a parameter
    - "Best guess" of the true value based on a sample
    - Single number – that's a problem…
    - P($\bar{X} = \mu$)? _OR_ P($\hat{p} = p$)?
    - Probability of a single point is ZERO!
    - Doesn't show "how close" the estimate is to the parameter
  - Confidence interval
    - Provides additional information about the variability of the estimate
    - Takes into account Margin of Error (MOE), or sampling error i.e., error due to chance

- **So… What is a Confidence Interval?**
  - **Confidence Interval (CI)** – interval containing the "**most believable**" values for a parameter
    - **Confidence level** – probability that this method produces an interval that contains (covers) the parameter and associated critical value (e.g., ***z*-score** or ***t*-score**)
    - **Confidence level** is usually close to 1.00 (most commonly 0.95 or 95%, but depends on **criticality** of the decision)
    - **Margin of Error –** measures how accurate the point estimate is likely to be
      - Multiples of the standard deviation (e.g. 1.96 * std dev)
  - Confidence Interval is constructed by using a **point estimate** and **adding** and **subtracting** the **margin of error** (that is, critical z-score times the standard error)

- That is, **CI = point estimate ± (Critical Value)(Std Error),** where:
  - Point Estimate is the sample statistic estimating the population parameter of interest
  - Critical Value is a table value based on the sampling distribution of the point estimate and the desired confidence level
  - Standard Error is the standard deviation of the point estimate
- How confident are we that the interval covers the unknown population parameter?
  - Some percentage (less than 100%)
  - 95% confident (probably most common), 99%, 90%
  - Desired level of confidence defines the “critical value” or z-score
  - **But that means, we are NEVER sure…**
- **Understanding Confidence Intervals**

Understanding Confidence Intervals

CI = point estimate ± (Critical Value)(Std Error)

**A 95% confidence interval is formed under the knowledge:

- 95% of all the possible intervals based on every possible sample from the population
- Would cover the parameter and the other 5% would miss

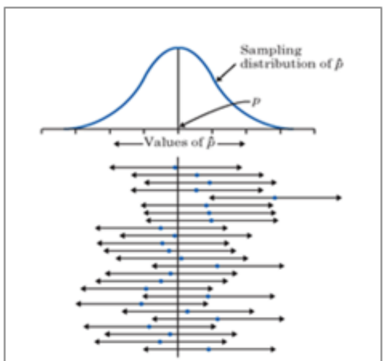


Figure 21.4 Twenty-five samples from the same population give these 95% confidence intervals. In the long run, 95% of all such intervals cover the true population proportion, marked by the vertical line. (*Statistics: Concepts and Controversies* (8th Edition), by Moore and Notz, W.H. Freeman and Company, 2013 p. 495)

- **Confidence Level, (1-α)**
  - Suppose confidence level = 95%
  - Also written (1 - α) = 0.95, (so α = 0.05)
  - A relative frequency interpretation: 95% of all the confidence intervals that can be constructed will contain the unknown true parameter
  - A specific interval either will contain or will not contain the true parameter
  - No **probability** involved in a specific interval

Confidence Level	Confidence Coefficient (1-α)	z-value
80%	0.80	1.28
90%	**0.90**	**1.645**
95%	**0.95**	**1.96**
98%	0.98	2.33
99%	**0.99**	**2.58**
99.80%	0.998	2.08
99.90%	0.999	3.27

Central Limit Theorem: Proportions AND Means

RULE: If many samples or repetitions of the SAME SIZE are taken, the frequency curve made from STATISTICS from the SAMPLES will be approximately normally distributed

Categorical (2 outcomes)

PROPORTIONS ($\hat{p}$'s):

- **Assumptions**:
  1. Population w/fixed proportion
  2. Random sample from population
  3. np≥5 and n(1-p)≥5 ("large" samples)
- **MEAN** of samples $\hat{p}$'s will be population proportion (p)

  ➔ $\mu_{\hat{p}} = p$
- **STANDARD DEVIATION** of the sample proportions ($\hat{p}$'s) will be:

  $\sigma_{\hat{p}} = \sqrt{\frac{(p)(1-p)}{n}}$

Quantitative (Measurement)

MEANS ($\bar{X}$'s):

- ***Conditions/Assumptions***
  1. If population bell-shaped (normal), random sample of any size
  2. If population not bell-shaped, a ***large*** random sample (≥ 30)
- **MEAN** of sample means ($\bar{X}$'s) will be population mean (μ)

  ➔ $\mu_{\bar{x}} = \mu$
- **STANDARD DEVIATION** of the sample means ($\bar{X}$'s) will be:

  $\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}}$

- **8.3 CIs for population proportion, *p*:**

$$\hat{p} \pm z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

- What are usual values of z?

Confidence Level	Error Probability	Z
.9	.10	1.645
.95	.05	1.96
.99	.01	2.58

- **Assumptions**:

- How can we achieve a narrower confidence interval?

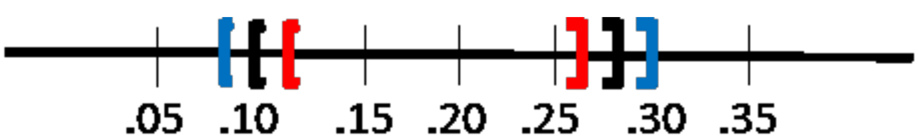
- **Example** 1: A planning committee needs to estimate the percentage of students at a large university who will attend an upcoming event so that they can determine an appropriate location for the event. 80 students are randomly selected, and 15 say that they will come to the event. What is a 95% confidence interval for the proportion of all the university's students who will attend the event?

  - 90% confidence interval?

  - 99% confidence interval?

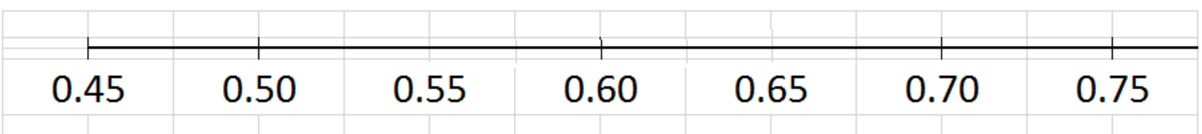
  - Interpret the interval.

  - Is the confidence interval a valid method for this problem?

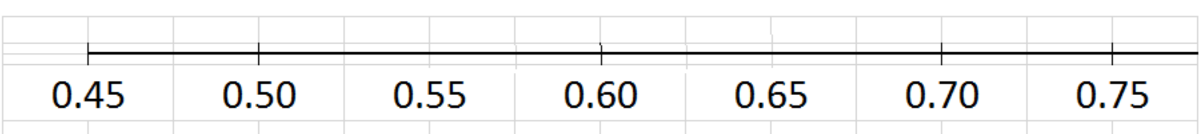


- **Example 2**: A random sample of 100 people shows that 25 are left-handed. Form a 95% confidence interval for the true proportion of left-handers.

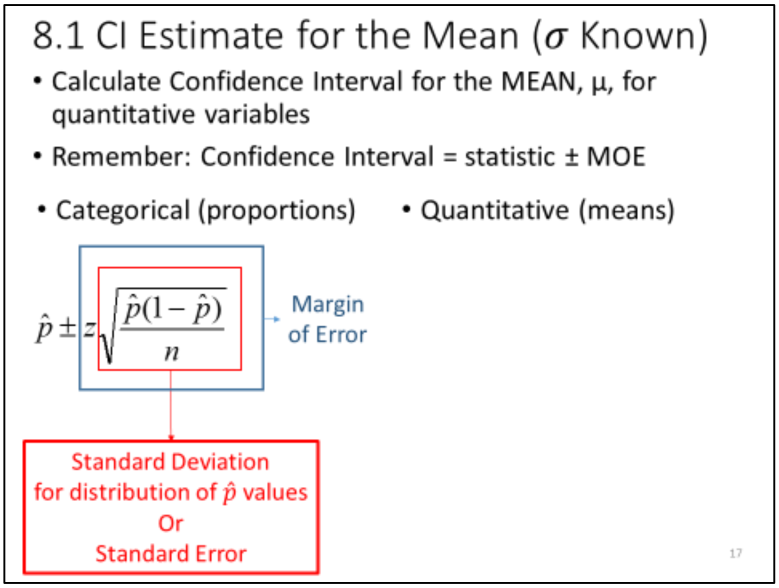
- **Example 3**: Source: Utts, ***Seeing Through Statistics***, : p. 384, #11: Contemplating switch from quarter system to semester system. Survey – random sample of ***n***=400 students. 240 prefer quarter system. Construct 95% Confidence Interval for true proportion who prefer to remain on the quarter system.
  - What do we know?
  - **Interpretation**?
  - Why can I say that more than half of the students prefer to stay on the quarter system versus semester? Look at the number line



  - But what if only 50 students had been surveyed and 30 preferred the quarter system? That is, n=50 and #successes=30



- **8.1 CI Estimate for the Mean (σ Known)**



  - **Example**: A sample of 11 circuits from a large normal population has a mean resistance of 2.20 ohms. We know from past testing that the population standard deviation is 0.35 ohms. Determine a 95% confidence interval for the true mean resistance of the population.

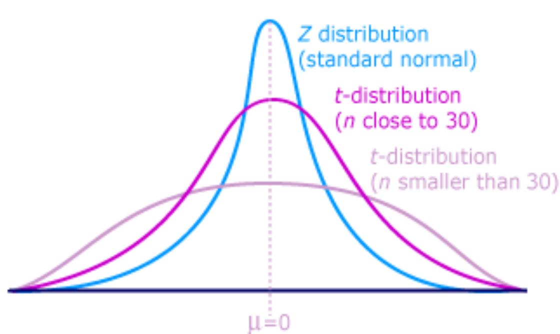
  - Interpretation?

- Can we ever know $\boldsymbol{\sigma}$?

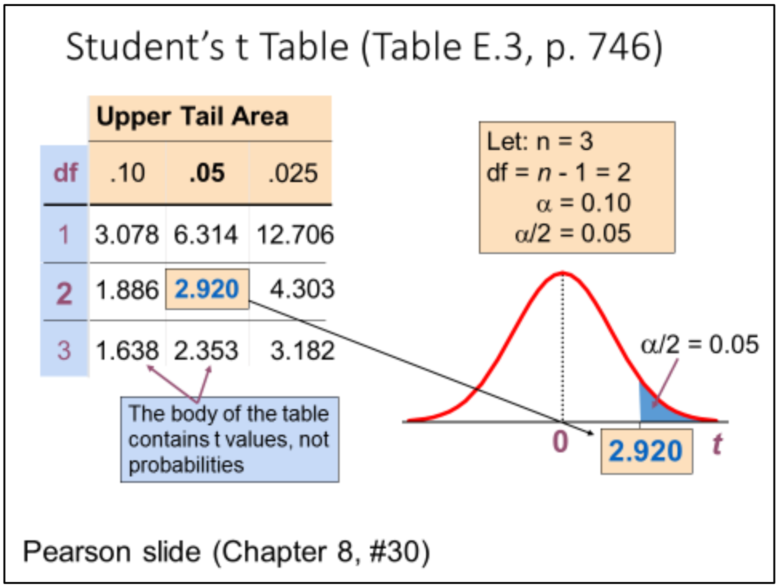
- **8.2 CI Estimate for the Mean (σ UNKnown)**
  - Student's ***t*** Distribution
    - William S. Gosset (under his pen name "Student")
    - Working for Guinness in Ireland
    - Trying to help brew better beer less expensively
    - Needed to make inferences about means without knowing $\boldsymbol{\sigma}$
  - Substitute the sample standard deviation, ***S***
  - Introduces extra uncertainty, since ***S*** is variable from sample to sample
  - Use the ***t*** distribution instead of the normal distribution

- ***t* - distribution**
  - Similar to normal distribution
    - Bell-shaped
    - Symmetric
    - Centered at mean/median/mode
    - BUT… tails are "heavier" ➔ more probability in the tails
    - Multiply by a little higher value (***t***-score) than z-score to get margin of error if ***n*** is small
  - Very close to z if ***n*** is large
  - DISADVANTAGES:
    - To use the ***t***-distribution, we must assume normality of the underlying population
    - **Degrees of freedom** (associated with sample size) = n-1

- **Degrees of Freedom**
  - **Idea**: Number of observations that are free to vary after sample mean has been calculated
  - For example: Suppose the mean of 3 numbers is 8.0
  - http://ci.columbia.edu/ci/premba_test/c0331/s7/s7_4.html



- Student’s t Table (Table E.3, p. 746)



- Selected t distribution values with comparison to the Z value

Confidence Level	t (10 d.f.)	t (20 d.f.)	t (30 d.f.)	Z (∞ d.f.)
0.80	1.372	1.325	1.310	1.28
0.90	1.812	1.725	1.697	1.645
0.95	2.228	2.086	2.042	1.96
0.99	3.169	2.845	2.750	2.58

- Note: t➔ Z as ***n*** increases

- **Confidence Interval for a Population Mean**
  - When the **standard deviation of the population is unknown**, the confidence interval for the population mean μ is

    CI = $\bar{x} \pm t\left(\frac{s}{\sqrt{n}}\right)$

  - “t-score” is based on the ***t***-distribution
    - Determined from the **level of confidence (α)** and

    - Degrees of freedom (df = ***n***–1), where ***n*** is the sample size

  - To use this method, you need:
    - Data obtained by **randomization**
    - **Approximately normal population** distribution
      - Especially important for small sample sizes (if non-normal, use large sample, i.e., ***n***>30)

      - Make a graphical display of the data and check for extreme outlier

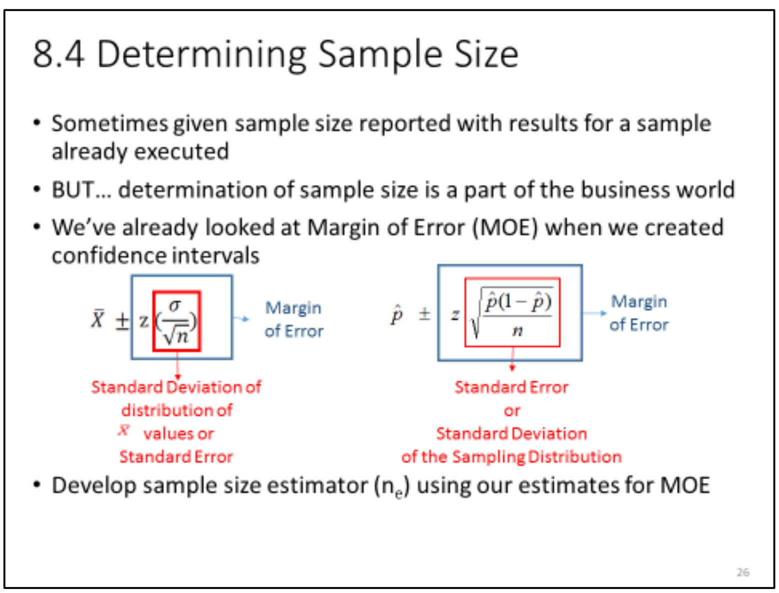
  - ***t***-distribution is a ***robust** method* in terms of the normality assumption

- **Example:** A random sample of ***n*** = 25 has $\overline{X} = 50$ and ***S*** = 8. Form a 95% confidence interval for μ.
  - What do we know?

  - Calculate CI.

  - Interpretation:

- **8.4 Determining Sample Size**



- **Sample size for means**
  - MOE is half the width of the interval
    - How wide do we want our interval to be? That is, what is the "appropriate" amount of sampling error?
    - What is our confidence level? 95%? 90? 99%? Other?
    - What is the estimate for σ, population standard deviation?
      - (Remember, haven't chosen the sample or calculated the point estimate yet…)
      - Use knowledge of the process or possibly a "pilot" study
  - $MOE = z(\frac{\sigma}{\sqrt{n_e}})$
- **Example:** If σ=45, what sample size is needed to estimate the mean within ±5 with 90% confidence?
  - What do we know?
  - $n_e = \frac{z^2\sigma^2}{(MOE)^2}$
  - ALWAYS round up

- **If σ is unknown**

- **Sample size for proportions**
  - MOE is half the width of the interval
    - How wide do we want our interval to be? That is, what is the "appropriate" amount of sampling error?
    - What is our confidence level? 95%? 90? 99%? Other?
    - What is the estimate for $p = P(success)$?
      - (Remember, haven't chosen the sample or calculated the point estimate yet…)
      - Use knowledge of the process or possibly a "pilot" study
  - $MOE = z\sqrt{\frac{p(1-p)}{n_e}}$

- **Example:** How large a sample would be necessary to estimate the true proportion of defective items in a large population within ±3%, with 95% confidence? (Assume a pilot sample yields $\hat{p}$= 0.12)
  - What do we know?
  - $n_e = \frac{(z)^2 p(1-p)}{(MOE)^2}$
  - ALWAYS round up

- **8.5 CI Estimation and Ethical Issues**
  - A confidence interval estimate (reflecting sampling error) should always be included when reporting a point estimate
  - The level of confidence should always be reported
  - The sample size should be reported
  - An interpretation of the confidence interval estimate should also be provided