STAT 511 Fall 2016 Midterm Exam Two (Practice) **Name:**____________________________

Ground Rules:

1. Exam 2 contains 6 questions. The maximum number of points on this exam is 100.
2. Exam 2 covers from Page 45 to Page 95 of the lecture notes.
3. Print your name at the top of this page in the upper right hand corner.
4. This is a closed-book and closed-notes exam. You may use a calculator if you wish, but SHOW ALL OF YOUR WORK AND EXPLAIN ALL OF YOUR REASONING!!!
5. Any discussion or ortherwise inappropriate communication between examinees, as well as the appearance of any unnecessary materials, will be dealt with severely.
6. Things that can only be used during the exam:
   - A TI-83 or equivalent calculator (no other electronic devices can be used)
   - Pen
   - Scratch papers that are provided along with the exam.
7. Keep 4 decimal places if needed.
8. You have 80 minuets to complete this exam. GOOD LUCK!

Honor Pledge for This Exam: After you have finished the exam, please read the following statement and sign your name below it.

I promise that I did not discuss any aspect of this exam with anyone other than the instructor, that I neither gave nor received any unauthorized assistance on this exam, and that the work presented herein is entirely my own.

Formula sheet (3 Pages)

The Binomial Expansion of $(x+y)^n$ Let x and y be any real numbers, then

$$(x+y)^n = \binom{n}{0} x^n y^0 + \binom{n}{1} x^{n-1} y^1 + \binom{n}{2} x^{n-2} y^2 + \cdots + \binom{n}{n} x^0 y^n$$
$$= \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i.$$

The Sum of a Geometric Series Let r be a real number such that $|r| < 1$, and m be any integer $m \geq 1$

$$\sum_{i=0}^{\infty} r^i = \frac{1}{1-r}, \quad \sum_{i=1}^{\infty} r^i = \frac{r}{1-r}, \quad \sum_{i=0}^{m} r^i = \frac{1-r^{m+1}}{1-r}.$$

The (Taylor) Series Expansion of e^x Let x be any real number, then

$$e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!}.$$

Some useful formulas for particular summations follow. The proofs (omitted) are most easily established by using mathematical induction.

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$
$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$
$$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

Gamma Function Let $t > 0$, then $\Gamma(t)$ is defined by the following integral:

$$\Gamma(t) = \int_0^{\infty} y^{t-1} e^{-y} dy.$$

Using the technique of integration by parts, it follows that for any $t > 0$

$$\Gamma(t+1) = t\Gamma(t)$$

and if $t = n$, where n is an integer,

$$\Gamma(n) = (n-1)!.$$

Further,

$$\Gamma(1/2) = \sqrt{\pi}.$$

If $\alpha, \beta > 0$, the ***Beta function***, $B(\alpha, \beta)$, is defined by the following integral,

$$B(\alpha, \beta) = \int_0^1 y^{\alpha-1}(1-y)^{\beta-1} dy$$

and is related to the gamma function as follows:

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

APPENDIX 2

Common Probability Distributions, Means, Variances, and Moment-Generating Functions

Table 1 Discrete Distributions

Distribution	Probability Function	Mean	Variance	Moment-Generating Function
Binomial	$p(y) = \binom{n}{y} p^y (1-p)^{n-y};$ $y = 0, 1, \ldots, n$	np	$np(1-p)$	$[pe^t + (1-p)]^n$
Geometric	$p(y) = p(1-p)^{y-1};$ $y = 1, 2, \ldots$	$\frac{1}{p}$	$\frac{1-p}{p^2}$	$\frac{pe^t}{1-(1-p)e^t}$
Hypergeometric	$p(y) = \frac{\binom{r}{y}\binom{N-r}{n-y}}{\binom{N}{n}};$ $y = 0, 1, \ldots, n$ if $n \le r$, $y = 0, 1, \ldots, r$ if $n > r$	$\frac{nr}{N}$	$n\left(\frac{r}{N}\right)\left(\frac{N-r}{N}\right)\left(\frac{N-n}{N-1}\right)$	does not exist in closed form
Poisson	$p(y) = \frac{\lambda^y e^{-\lambda}}{y!};$ $y = 0, 1, 2, \ldots$	λ	λ	$\exp[\lambda(e^t - 1)]$
Negative binomial	$p(y) = \binom{y-1}{r-1} p^r (1-p)^{y-r};$ $y = r, r+1, \ldots$	$\frac{r}{p}$	$\frac{r(1-p)}{p^2}$	$\left[\frac{pe^t}{1-(1-p)e^t}\right]^r$

Table 2 Continuous Distributions

Distribution	Probability Function	Mean	Variance	Moment-Generating Function
Uniform	$f(y) = \frac{1}{\theta_2 - \theta_1}; \theta_1 \le y \le \theta_2$	$\frac{\theta_1 + \theta_2}{2}$	$\frac{(\theta_2 - \theta_1)^2}{12}$	$\frac{e^{t\theta_2} - e^{t\theta_1}}{t(\theta_2 - \theta_1)}$
Normal	$f(y) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left[-\left(\frac{1}{2\sigma^2}\right)(y-\mu)^2\right]$ $-\infty < y < +\infty$	μ	σ^2	$\exp\left(\mu t + \frac{t^2\sigma^2}{2}\right)$
Exponential	$f(y) = \frac{1}{\beta}e^{-y/\beta}; \quad \beta > 0$ $0 < y < \infty$	β	β^2	$(1 - \beta t)^{-1}$
Gamma	$f(y) = \left[\frac{1}{\Gamma(\alpha)\beta^\alpha}\right] y^{\alpha-1}e^{-y/\beta};$ $0 < y < \infty$	$\alpha\beta$	$\alpha\beta^2$	$(1 - \beta t)^{-\alpha}$
Chi-square	$f(y) = \frac{(y)^{(v/2)-1}e^{-y/2}}{2^{v/2}\Gamma(v/2)};$ $y > 0$	v	$2v$	$(1 - 2t)^{-v/2}$
Beta	$f(y) = \left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right] y^{\alpha-1}(1-y)^{\beta-1};$ $0 < y < 1$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	does not exist in closed form

Question 1. Suppose that Y has a geometric distribution with success probability p. Let

$$F_Y(y) = P(Y \leq y)$$

denote the cumulative distribution function of Y.

(a) Show that $F_Y(a) = 1 - (1-p)^a$ for any positive integer a.

(b) Using the results in (a), argue that the geometric distribution possesses the **memoryless property**; i.e., show that for integer $b > 0$,

$$P(Y > a + b \mid Y > a) = P(Y > b).$$

(c) Show the mgf of Y is given by

$$m_Y(t) = \frac{pe^t}{1 - qe^t},$$

where $q = 1 - p$, for $t < -\ln q$. Make sure you argue why $t < -\ln q$.

Question 2. Past studies have shown that 1 out of every 10 cars on the road has a speedometer that is miscalibrated. For this problem, assume that different cars are independent and that each has the same 1/10 probability of being miscalibrated.

(a) Suppose that we continually observe cars until we find the first car with a miscalibrated speedometer. Let Y denote the number of cars that we will observe. Write the pmf for Y and computer $P(Y \leq 6)$. Further find the mean and variance of Y.

(b) Suppose that we continually observe cars untile we find the second car with a miscalibrated speedometer. Let Z denote the number of cars that we will observe. Identify the distribution of Z, compute $P(Z = 4)$, $P(Z \leq 3)$, and find the mean and variance of Z.

Question 3. In my fish tank at home, there are 20 fish. Seven have been tagged. If I catch five fish (and random, and without replacement), let Y denote the number of tagged fish that I have caught. Identify the distribution of Y, compute $P(Y = 2)$, $P(Y \leq 3)$, and find the mean and variance of Y.

Question 4. Suppose that Y, the number of infected trees distributed on a square-acre plot, follows (approximately) a Poisson distribution with mean $\lambda = 3$.

(a) What is the probability a single square-acre plot will contain 2 or fewer infected trees?

(b) The time, T, (measured in hours) needed to locate and treat all infected trees is thought to be a linear function of Y. One researcher assumes that $T = 2Y + 3$. Find the mean and variance of T.

(c) Suppose that we continually observe square-acre plots (in a very large forest) until we observe the fourth plot with 2 or fewer infected trees. Let X denoted the number of plots we will need to observe. Write down the pmf for X and compute $P(X = 6)$. You many assume that the square-acre plots are independent.

Question 5. Suppose that the random variable Y has the pdf

$$f_Y(y) = \begin{cases} cy, & 1 < y < 5 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the value of c which makes this a valid pdf, and grad this pdf. Is this a named distribution? If so, which one is it?

(b) Find the cumulative distribution function $F_Y(y)$. Remember that the cdf is defined for all $-\infty < y < \infty$. Graph the cdf.

(c) Compute $E(Y)$ and $V(Y)$ without using the mfg.

(d) Derive the mgf of Y.

(e) Find $P(2 < Y < 4)$.

Note: I want very detailed graphs.

Question 6. In the article "Modelling sediment and water column interactions for hydrophobic pollutants" (*Water Research*, **1984**, 1169-1174), the authors model sediment density in a particular region (Y, measured in g/cm) as a normal random variable with mean 2.65 and standard deviation 0.85.

(a) In this region, what is the probability that a sediment specimen exceeds 3.80 g/cm?

(b) Suppose you do not know the value of the mean of Y, but you know the standard deviation is 0.85 and the probability $P(Y > 3.8) = 0.05$. Find the mean of Y.

(c) Suppose you do not know the value of the standard deviation of Y, but you know the mean is 2.65 and the probability $P(Y > 3.8) = 0.05$. Find the standard deviation of Y.

Question 7. For a certain class of new jet engines, the time (in years) until an overhaul is needed varies according to the following probability density function:

$$f_Y(y) = \begin{cases} \frac{1}{4}e^{-y/4}, & y > 0 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Is this a named distribution? If so, what is it?

(b) Derive the cumulative distribution function of Y.

(c) Find the probability that one of these engines will need an overhaul during its first year.

(d) Find $E(Y)$ and $V(Y)$.

(e) Find the mgf $m_Y(t)$.

(f) Find $P(Y > 4|Y > 1)$.

Question 8. An environmental engineer at a large gravel company models Y, the monthly gravel sales (in thousands of tons) as a random variable with pdf

$$f_Y(y) = \begin{cases} c(1-y)^3, & 0 < y < 1 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the value of c.

(b) Is this a named distribution? If so, what is it?

(c) Compute $E(Y)$ and $V(Y)$.

(d) Find $P(Y < 0.3)$.

(e) Find the median, $\phi_{0.5}$, of this distribution.

(f) Suppose that the monthly revenue (in \$100,000) from gravel sales is given by $R = 10(1 - Y)$. Find the mean and standard deviation of R. What is the probability that the monthly revenue exceeds \$500,000?

Question 9. In a toxicology experiment, Y denotes the death time (in minutes) for a single rat treated with a certain toxin. The pdf for Y is given by

$$f_Y(y) = \begin{cases} cye^{-y/4}, & y > 0 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the value of c that makes this a vlid pdf.

(b) Is this a named distribution? If so, what is it?

(c) Find $E(Y)$ and $V(Y)$.

(d) Find $P(Y < 5)$ and $P(Y = 5)$.

(e) Find the mgf $m_Y(t)$ of Y.

Question 10. Explosive devices used in mining operations produce (nearly) circular craters when detonated. The radius of these craters, say R, follows a uniform distribution with $\theta_1 = 0$ and θ_2 unknown. The area of the crater is $Y = \pi R^2$. Suppose $E(R) = 20$.

(a) Find the value of θ_2.

(b) Find the mean of Y.

(c) Find the probability that $P(Y > 100\pi)$.