1. During one week, Joe will receive 7 text messages.
(a) What is the probability that Joe will receive exactly one text message each day? State any assumptions you need to answer this question. I would start by characterizing the underlying sample space.
(b) If 30 percent of all of Joe's text messages are from Polly, find the probability that at least 2 of the 7 text messages during one week are from Polly. State any assumptions you need to answer this question.

(a) I assume every outcome occurs with the equal chance

the probability is $\frac{7!}{7^7}$

(b) Assumption: text are independent

the chance of "each text is from Polly" is 0.3

X: # of text messages from Polly

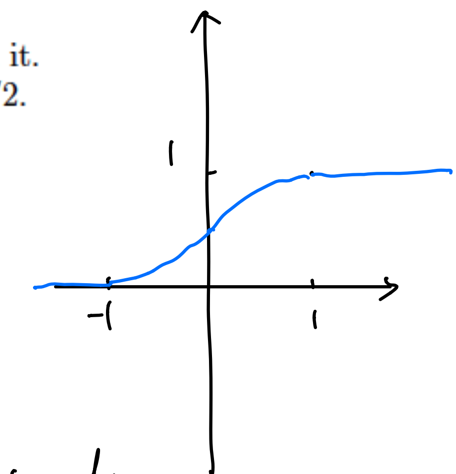
$X \sim$ Binomial $(n=7, p=0.3)$

$$P(X \geq 2) = 1 - P(X<2)$$
$$= 1 - P(X \leq 1)$$
$$= 1 - \text{binomcdf}(\cdots\cdots)$$

2. A normalized measurement of color for automotive paint is always guaranteed to fall between -1 and 1. Specifically, the measurement Y is a random variable with pdf

$$f_Y(y) = \begin{cases} \frac{3}{4}(1-y^2), & -1 < y < 1 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the cumulative distribution function of Y and graph it.
(b) Find the probability that Y is greater than or equal to $1/2$.



(a) for $y \le -1$, $F_Y(y) = 0$

for $y \geq 1$, $F_Y(y) = 1$

for $-1 < y < 1$, $F_Y(y) = \int_{-1}^{y} f_Y(u)\,du$

$$= \int_{-1}^{y} \frac{3}{4}(1-u^2)\,du = \left(\frac{3}{4}u - \frac{1}{4}u^3\right)\Big|_{-1}^{y}$$

$$= \frac{3}{4}y - \frac{1}{4}y^3 + \frac{3}{4} - \frac{1}{4} = \frac{1}{4}(3y+2-y^3)$$

(b) $P(Y \geq \frac{1}{2}) = 1 - P(Y < \frac{1}{2}) = 1 - F(\frac{1}{2})$

$$= 1 - \frac{1}{4}\left(3 \times \frac{1}{2} + 2 - \left(\frac{1}{2}\right)^3\right)$$

3. Suppose that Y is a random variable with mean $\mu = E(Y)$, variance $\sigma^2 = V(Y)$, and moment-generating function $m_Y(t)$. Define $Z = a + bY$, where a and b are constants.
(a) Show that $E(Z) = a + b\mu$.
(b) Show that $V(Z) = b^2\sigma^2$.
(c) Show that $m_Z(t) = e^{at} m_Y(bt)$.

(a) $E(Z) = E[a+bY] = a + bE[Y] = a + b\mu$

(b) $V(Z) = V[a+bY] = b^2 V(Y) = b^2 \sigma^2$

(c) $M_Z(t) = E[e^{tZ}] = E[e^{t(a+bY)}]$

$= E[e^{ta} \times e^{(tb)Y}]$

$= e^{ta} E(e^{(tb)Y})$

$= e^{ta} m_Y(tb)$

4. A purchaser of electrical components buys them in lots of size 10 from two different suppliers. It is her policy to inspect 3 components randomly from the lot and to accept the lot only if all 3 are nondefective.

- The purchaser buys 30 percent of her lots from Supplier 1; Supplier 1 lots always contain 4 defectives out of 10.
- The purchaser buys 70 percent of her lots from Supplier 2; Supplier 2 lots always contain 1 defective out of 10.

What is the probability that the next lot will be accepted?
Hint: Define A to be the event that the lot is accepted, B_1 to be the event that the next lot is from Supplier 1, and B_2 to be the event that the next lot is from Supplier 2. There are only 2 suppliers, so $\{B_1, B_2\}$ partitions the space of possible suppliers. You want to compute $P(A)$.

$$P(A) = P(A \mid B_1)\,P(B_1) + P(A \mid B_2)\,P(B_2)$$

where $P(B_1) = 30\%$, $\quad P(B_2) = 70\%$

$$P(A \mid B_1) = \frac{\binom{4}{0}\binom{6}{3}}{\binom{10}{3}}$$

$$P(A \mid B_2) = \frac{\binom{1}{0}\binom{9}{3}}{\binom{10}{3}}$$

5. The lifetime Y (in 1000s of hours) of a certain type of 100-watt industrial strength light bulb is a random variable with pdf

$$f_Y(y) = \begin{cases} \frac{1}{2}e^{-y/2}, & 0 < y < \infty \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the probability that one randomly selected lightbulb will last longer than 3000 hours; that is, compute $P(Y > 3)$.
(b) Suppose that we continue observing lightbulbs until we find the **first** one whose lifetime exceeds 3000 hours. What is the probability that we will need to observe **no more than** 4 lightbulbs?
(c) Suppose that we continue observing lightbulbs until we find the **third** one whose lifetime exceeds 3000 hours. What is the probability that we will have to observe **at least** 5 lightbulbs?
Note: In parts (b) and (c), assume that all lightbulbs are independent with the same probability from part (a).

(a) $P(Y>3) = \int_3^{+\infty} \frac{1}{2} e^{-y/2}\, dy = \left(-e^{-y/2}\right)\Big|_3^{+\infty} = e^{-3/2}$

(b) Let X be the # of lightbulbs until the first one whose lifetime exceeds 3000 hours

$X \sim \text{Geometric}\ (e^{-3/2})$

$P(X \leq 4) = \text{geometric cdf}\ (\ ...\ )$

(c) Let Z be the # of .... until the 3rd ....

$Z \sim \text{nib}\ (r=3,\ e^{-3/2})$

$P(Z \geq 5) = 1 - P(Z < 5)$

$= 1 - P(Z \leq 4)$

$= \ ....$

6. Suppose that Y is a random variable with mean μ and variance σ^2. Recall that the skewness and kurtosis for a random variable Y are defined as

$$\xi = E[(Y-\mu)^3]/\sigma^3 \quad \text{(skewness)},$$

and

$$\kappa = E[(Y-\mu)^4]/\sigma^4 \quad \text{(kurtosis)},$$

variability asymmetry

respectively.
(a) If Y is a **standard normal** random variable; i.e., $Y \sim \mathcal{N}(0,1)$, show mathematically that $\xi = 0$ and $\kappa = 3$.

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not important → (b) For any random variable, describe to me, in words, what μ, σ^2, ξ, and κ measure. You can be brief.

(a) $$\xi = \frac{E[(Y-\mu)^3]}{\sigma^3} = E\left(\frac{(Y-\mu)^3}{\sigma^3}\right) = E\left[\left(\frac{Y-\mu}{\sigma}\right)^3\right]$$

$$K = E\left[\left(\frac{Y-\mu}{\sigma}\right)^4\right]$$

Let $Z = \frac{Y-\mu}{\sigma}$, $Z \sim N(0,1)$ then $\xi = E[Z^3]$, $K = E[Z^4]$

$$M_Z(t) = \exp\left(\frac{t^2}{2}\right)$$

$$M_Z'(t) = t\exp\left(\frac{t^2}{2}\right)$$

$$M_Z''(t) = \exp\left(\frac{t^2}{2}\right) + t^2\exp\left(\frac{t^2}{2}\right)$$

$$M_Z^{(3)}(t) = t\exp\left(\frac{t^2}{2}\right) + 2t\exp\left(\frac{t^2}{2}\right) + t^3\exp\left(\frac{t^2}{2}\right)$$

$$M_Z^{(4)}(t) = \exp\left(\frac{t^2}{2}\right) + t^2\exp\left(\frac{t^2}{2}\right) + 2\exp\left(\frac{t^2}{2}\right) + 2t^2\exp\left(\frac{t^2}{2}\right) + 3t^2\exp\left(\frac{t^2}{2}\right) + t^4\exp\left(\frac{t^2}{2}\right)$$

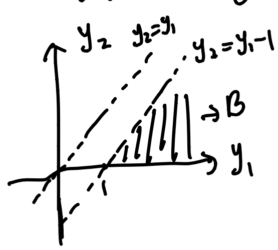
$E[Z^3] = M_Z^{(3)}(0) = 0$ $\quad$ $E(Z^4) = M_Z^{(4)}(0) = 3$

7. The management at a fast-food outlet is interested in the joint behavior of the random variables Y_1 and Y_2. The variable Y_1 denotes the total time (in minutes) between a customers arrival at the store and his/her departure from the service window. The variable Y_2 denotes the time (in minutes) a customer waits in line before reaching the service window. Both Y_1 and Y_2 are measured in minutes. The joint distribution of Y_1 and Y_2 is given by

$$f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} e^{-y_1}, & 0 < y_2 < y_1 < \infty \\ 0, & \text{otherwise.} \end{cases}$$

(a) Compute the probability that $Y_1 - Y_2$, the time spent at the service window, is greater than 1 minute; that is, compute $P(Y_1 - Y_2 > 1)$.
(b) Find both marginal distributions.
(c) Find the conditional distribution of Y_1 for customers whose value of $Y_2 = y_2 = 2.5$.

(a) $B = \{(y_1, y_2):\ 0 < y_2 < y_1 < +\infty,\ y_1 - y_2 > 1\}$



$$P(Y_1 - Y_2 > 1) = \underline{\int_1^{+\infty}\int_0^{y_1-1}} e^{-y_1}\, dy_2\, dy_1$$

$$= \int_1^{+\infty} e^{-y_1}(y_1 - 1)\, dy_1 \qquad \text{Let } u = y_1 - 1 \quad y_1 = u+1$$
$$du = dy_1$$

$$= \int_0^{+\infty} e^{-(u+1)}\, u\, du$$

$$= e^{-1} \underbrace{\int_0^{+\infty} e^{-u}\, u\, du}_{\text{Mean of } Exp(1)} = e^{-1} \times 1 = e^{-1}$$

or $\Gamma(\alpha)\beta^{\alpha} = \int_0^{+\infty} e^{-y/\beta} \times y^{\alpha-1}\, dy$

$\to \Gamma(2)1^2 = 1$

(b) $f_{Y_1}(y_1) = \int_0^{y_1} e^{-y_1} dy_2 = y_1 e^{-y_1}, \quad 0 < y_1 < +\infty$

$f_{Y_2}(y_2) = \int_{y_2}^{+\infty} e^{-y_1} dy_1 = \left(-e^{-y_1}\right)\Big|_{y_2}^{+\infty} = e^{-y_2}, \quad 0 < y_2 < +\infty$

Note: $Y_1 \sim \text{Gamma}(2, 1)$

$Y_2 \sim \text{Exp}(1)$

(c) $$f_{Y_1|Y_2}(y_1|y_2) = \frac{f_{Y_1,Y_2}(y_1, y_2)}{f_{Y_2}(y_2)} = \frac{e^{-y_1}}{e^{-y_2}}$$

$$= e^{-(y_1 - y_2)}, \quad y_2 < y_1 < +\infty$$

So given $Y_2 = y_2 = 2.5$

the density of Y_1 is $f_{Y_1|Y_2=2.5}(y_1|2.5) = e^{-(y_1-2.5)}$,

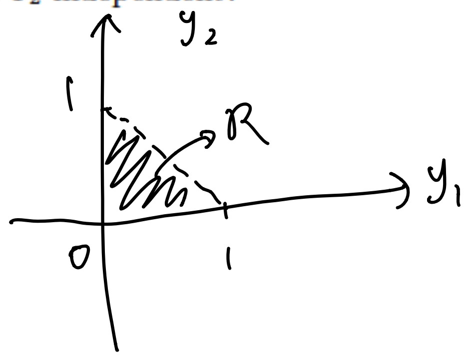
$2.5 < y_1 < +\infty$

8. Let Y_1 and Y_2 denote the proportions of two different chemicals in a mixture of chemicals used as an insecticide. Suppose that (Y_1, Y_2) has the joint probability density function

$$f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} 6y_1, & 0 \le y_1 \le 1,\ 0 \le y_2 \le 1,\ 0 \le y_1 + y_2 \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

(a) The set $R = \{(y_1, y_2) : 0 \le y_1 \le 1,\ 0 \le y_2 \le 1,\ 0 \le y_1 + y_2 \le 1\}$ is the two-dimensional support set of (Y_1, Y_2). Sketch a picture of R.
(b) Describe what $f_{Y_1,Y_2}(y_1, y_2)$ looks like geometrically.
(c) Compute $\mathrm{Cov}(Y_1, Y_2)$.
(d) Are Y_1 and Y_2 independent?

(a)



① $f_{Y_1}(y_1)$

$E(Y_1) = \int y_1 f_{Y_1}(y_1)\, dy_1$

(b) a triangular

(c) $Cov(Y_1, Y_2) = E(Y_1 Y_2) - E(Y_1)E(Y_2)$

$$E(Y_1Y_2) = \int_0^1 \int_0^{1-y_1} y_1 y_2 \times 6y_1 \, dy_2\, dy_1$$

$g(y_1, y_2) = y_1 \times y_2$

$E[g(Y_1, Y_2)] = \iint g(Y_1, Y_2) f(Y_1, Y_2)\, dy_1 dy_2$

$$= \int_0^1 6y_1^2 \left(\frac{1}{2} y_2^2 \Big|_0^{1-y_1} \right) dy_1$$

$$= \int_0^1 3y_1^2 (1-y_1)^2 \, dy_1 = 3 \times \frac{\Gamma(3)\Gamma(3)}{\Gamma(6)} = 3 \times \frac{2! \times 2!}{5!}$$

$$= \frac{1}{10}$$

$$\boxed{\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} = \int_0^1 y^{\alpha-1}(1-y)^{\beta-1}\, dy}$$

$g(y_1, y_2) = y_1$

$$E(Y_1) = \int_0^1 \int_0^{1-y_1} y_1 \times 6y_1 \, dy_2 \, dy_1$$

$$= \int_0^1 6y_1^2 (1-y_1) dy_1$$

$$= 6 \times \frac{\Gamma(3)\Gamma(2)}{\Gamma(5)} = 6 \times \frac{2! \times 1}{4!} = \frac{1}{2}$$

$$E(Y_2) = \int_0^1 \int_0^{1-y_1} y_2 \, 6y_1 \, dy_2 \, dy_1$$

$$= \int_0^1 6y_1 \left(\frac{1}{2} y_2^2 \Big|_0^{1-y_1}\right) dy_1$$

$$= \int_0^1 3y_1(1-y_1)^2 dy_1$$

$$= 3 \times \frac{\Gamma(2)\Gamma(3)}{\Gamma(5)} = 3 \times \frac{2!}{4!} = \frac{1}{4}$$

$$Cov(Y_1, Y_2) = \frac{1}{10} - \frac{1}{2} \times \frac{1}{4} = \frac{1}{10} - \frac{1}{8} = -\frac{1}{40}$$

(d) No, if Y_1 and Y_2 are independent, $Cov(Y_1, Y_2) = 0$

but $Cov(Y_1, Y_2) = -\frac{1}{40} \neq 0$

So not independent