Old Exam 1, a fifty min version

Monday, September 19, 2016 3:42 PM

Old Exam 1,
a fifty min...

Tentative Solution!
If I did anything wrong, let me know. Thanks

GROUND RULES:

- This exam contains 5 questions; each question is worth 10 points. The maximum number of points on this exam is 50.
- Print your name **at the top of this page in the upper right hand corner**.
- This is a closed-book and closed-notes exam. You may use a calculator if you wish, but SHOW ALL OF YOUR WORK AND EXPLAIN ALL OF YOUR REASONING!!!
- Any discussion or otherwise inappropriate communication between examinees, as well as the appearance of any unnecessary material, will be dealt with severely.
- You have 60 minutes to complete this exam. GOOD LUCK!

HONOR PLEDGE FOR THIS EXAM:

After you have finished the exam, please read the following statement and sign your name below it.

> *I promise that I did not discuss any aspect of this exam with anyone other than the instructor, that I neither gave nor received any unauthorized assistance on this exam, and that the work presented herein is entirely my own.*

1. Suppose that S is a non-empty sample space and that A and B are subsets of S with $P(A) = 0.5$, $P(B) = 0.6$, and $P(A \cup B) = 0.8$.

(a) Are A and B independent? Prove/explain your answer.
(b) Are A and B mutually exclusive? Prove/explain your answer.
(c) Find $P(A|B)$.
(d) Find the probability that A occurs or B occurs, but not both.

(a) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$.8 = .5 + .6 - P(A \cap B)$

$P(A \cap B) = 0.3$

Since $P(A \cap B) = P(A) \times P(B)$, A and B are independent

(b) No, $P(A \cap B) \neq 0$

(c) $P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{0.3}{0.6} = 0.5$

(d) Note that $(A \cap \bar{B})$ and $(\bar{A} \cap B)$ are mutually exclusive

$P((A \cap \bar{B}) \cup (\bar{A} \cap B)) = P(A \cap \bar{B}) + P(\bar{A} \cap B)$

$= P(A) - P(A \cap B) + P(B) - P(A \cap B)$

$= 0.5 - 0.3 + 0.6 - 0.3$

$= 0.5$

2. Clinical trials are underway to get an H1N1 (swine flu) vaccination to the public as quickly as possible. In one trial at Vanderbilt University Hospital, patients are randomly assigned to one of the four treatment groups:

1. Placebo (a non-treatment, for control purposes)
2. Experimental 1: 7.5 mcg of hemagglutinin
3. Experimental 2: 15 mcg of hemagglutinin
4. Experimental 3: 30 mcg of hemagglutinin.

There are 200 patients in the trial; 20 are randomly assigned to the Placebo group and 60 are randomly assigned to each of the experimental groups. Patients will be monitored for 6 months to see if they contract H1N1. Physicians in charge of the trial have conjectured that

- 15 percent of the Placebo patients will contract H1N1 in 6 months
- 8 percent of the Experimental 1 patients will contract H1N1 in 6 months
- 4 percent of the Experimental 2 patients will contract H1N1 in 6 months
- 1 percent of the Experimental 3 patients will contract H1N1 in 6 months

(a) Suppose that a patient in the trial is selected at random. What is the probability that the patient contracts H1N1 within 6 months?
(b) Suppose that a patient contracts H1N1 within 6 months. What is the probability that this patient was assigned to the Placebo group?
(c) Suppose that a patient does not contract H1N1 within 6 months. What is the probability that this patient was assigned to the Experimental 3 group?

Your solutions can be written on the next page if you need additional space.

(a) Denote "a patient contracts H1N1 within 6 month" by A
Denote the patient is in the group of Placebo by E0
- - - - - - - - - - - - of Experiment 1 by E1
- - - - - - - - - - - - - - 2 by E2
- - - - - - - - 3 by E3

This is an extra page for Problem 2.

$$P(A) = P(A|E0)P(E0) + P(A|E1)P(E1) + P(A|E2)P(E2) + P(A|E3)P(E3)$$

$$= 0.15 \times \frac{20}{200} + 0.08 \times \frac{60}{200} + 0.04 \times \frac{60}{200} + 0.01 \times \frac{60}{200}$$

$$= \frac{3+4.8+2.4+0.6}{200} = \frac{10.8}{200} = 0.054$$

(b) $$P(E0|A) = \frac{P(E0 \cap A)}{P(A)} = \frac{P(A|E0)P(E0)}{P(A)}$$

$$= \frac{0.15 \times \frac{20}{200}}{0.054}$$

$$= \frac{0.015}{0.054} \approx 0.2778$$

(c) $$P(E3|\bar{A}) = \frac{P(E3 \cap \bar{A})}{P(\bar{A})} = \frac{P(\bar{A}|E3)P(E3)}{P(\bar{A})}$$

$$= \frac{[1 - P(A|E3)]\, P(E3)}{1 - P(A)}$$

$$= \frac{(1-0.01) \times \frac{60}{200}}{1-0.054} \approx 0.3140$$

3. A project manager has 5 chemical engineers on her staff: 2 are women and 3 are men. The engineers are equally qualified. Consider the random experiment of choosing 2 engineers for 2 assignments (1 engineer per assignment).

Note: For notational purposes, it might be easiest to refer to the engineers as w_1, w_2, m_1, m_2, and m_3.

(a) If the assignments are identical (so that order of selection is not important), write out the sample space for this experiment.
(b) Define A to be the event that no women are chosen. Under the assumption that all sample points are equally likely, find $P(A)$.
(c) For this part only, assume that the 2 assignments are distinct (e.g., the assignments are in different regions of the US). Under this assumption, the sample space contains how many sample points?

(a) $S = \{ (w_1, m_1), (w_1, m_2), (w_1, m_3), (w_2, m_1), (w_2, m_2), (w_2, m_3), (w_1, w_2), (m_1, m_2), (m_1, m_3), (m_2, m_3) \}$

(b) $P(A) = P\big((m_1, m_2) \text{ or } (m_1, m_3), \text{ or } (m_2, m_3)\big)$

$= \frac{3}{10}$

(c) Order matters, permutation:

$P_{5,2} = \frac{5!}{(5-2)!} = 5 \times 4 = 20$

4. The number of claims per week at an insurance company is a random variable Y. Suppose that Y has probability mass function

$$p_Y(y) = \begin{cases} ce^{-2y}, & y = 0, 1, 2, \ldots \\ 0, & \text{otherwise.} \end{cases}$$

The moment generating function (mgf) of Y is given by

$$m_Y(t) = c(1 - e^{t-2})^{-1},$$

for values of $t < 2$. **You do not need to prove this.**

(a) Show that $c = 1 - e^{-2}$.
(b) What is the probability that there are at least 2 claims in a given week?
(c) Find $E(Y)$.

(a) $m_Y(t) = E[e^{tY}] = c(1-e^{t-2})^{-1}$

Taking $t=0$ leads to $E[e^{0Y}] = c(1-e^{0-2})^{-1}$

Since $E[e^{0Y}] = E[1] = 1$,

$1 = c(1-e^{-2})^{-1} \Rightarrow c = 1-e^{-2}$

(b) $P(Y \geq 2) = 1 - P(Y=0) - P(Y=1)$

$= 1 - ce^{-2\times 0} - ce^{-2\times 1}$

$= 1 - c(1+e^{-2}) = 1-(1-e^{-4}) = e^{-4}$

(c) $E(Y) = \left.\frac{dm_Y(t)}{dt}\right|_{t=0} = m_Y'(0)$ $\quad m_Y(t) = (1-e^{-2})(1-e^{t-2})^{-1}$

$m_Y'(t) = (1-e^{-2})(-1)(1-e^{t-2})^{-1-1}(1-e^{t-2})'$

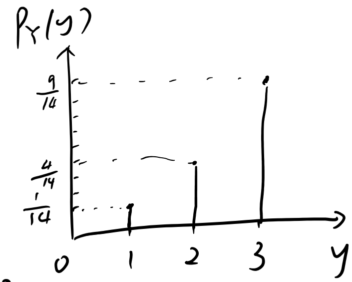
$= (1-e^{-2})(-1)(1-e^{t-2})^{-2}(-e^{t-2})$

$= (1-e^{-2})(1-e^{t-2})^{-2}e^{t-2}$, $E(Y) = m_Y'(0) = (1-e^{-2})^{-1}e^{-2} \approx 0.1565$

5. Let Y be a random variable with pmf

$$p_Y(y) = \begin{cases} y^2/14, & y = 1, 2, 3 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Graph the probability histogram for Y. Label your axes.
(b) Find $E(Y)$ and $V(Y)$.
(c) Find $E(2Y^2 - 6Y + 3)$.
(d) Find $V(4Y - 3)$.



(b) $E[Y] = 1 \times \frac{1^2}{14} + 2 \times \frac{2^2}{14} + 3 \times \frac{3^2}{14}$

$= \frac{1+8+27}{14} \simeq 2.5714$

$V[Y] = E[Y^2] - (E[Y])^2$

$= 1^2 \times \frac{1^2}{14} + 2^2 \times \frac{2^2}{14} + 3^2 \times \frac{3^2}{14} - (2.5714)^2$

$= \frac{1+16+81}{14} - (2.5714)^2 = 7 - (2.5714)^2$

≈ 0.3879

(c) $E(2Y^2 - 6Y + 3) = E(2Y^2) - E(6Y) + E[3]$

$= 2E(Y^2) - 6E(Y) + 3$

$= 2 \times 7 - 6 \times 2.5714 + 3$

$= 1.5716$

(d) $V(4Y-3) = 4^2 V(Y) = 16 \times 0.3879 = 6.2064$