CHAPTER 2 PROBLEMS

1. On the first day of class, we discussed the underlying notion of a random experiment, an associated sample space, and assigning probabilities to events in the sample space. Describe a phenomena that will happen to you soon (e.g., this weekend, this year, etc.) and conceptualize it as a random experiment. Describe your associated sample space, events of interest to which you might like to assign probability (using appropriate set notation), and a procedure for assigning probabilities to events. I'm looking for creativity!

2. Suppose that we have n identical white balls, numbered $1, 2, ..., n$, in one drum. Suppose that we have m identical red balls, numbered $1, 2, ..., m$, in a second drum. Consider the following experiment of

- choosing r balls from the first drum and observing the ball numbers, AND
- choosing s balls from the second drum and observing the ball numbers.

That is, balls are drawn from each drum. In addition, balls are drawn from each drum at random and **without replacement**; i.e., balls are not replaced after they are selected.
(a) Describe a sample space for this experiment, which, within color, does not regard the ordering of the balls drawn as important. Each sample point should be a vector of length $r + s$, corresponding to the $r + s$ numbers chosen.
(b) Suppose that we pick a sample point at random from the underlying sample space. What is the probability associated with this point? What assumptions are you making?
(c) Evaluate your expression in part (b) when $r = 5$, $n = 55$, $s = 1$, and $m = 42$. If your answer in part (b) is correct, your answer here is the probability of winning the Powerball lottery!

3. Suppose that an experiment is to be performed where the sample space $S = (0, 1)$ and the probability measure P assigns probabilities to events $A \subset S$ using the rule

$$P(A) = \int_A f(y)dy,$$

where $f(y) = 6y(1 - y)$, for $0 < y < 1$, and $f(y) = 0$, otherwise. Prove that the probability measure P satisfies the three Kolmolgorov axioms.

4. There were seven accidents in a town during a seven-day period. Define A to be the event that all seven accidents occurred on the same day. Define B to be the event that each of the seven accidents occurred on a different day. Find $P(A)$ and $P(B)$. What assumptions are you making?

5. An insurance company examines its pool of auto insurance customers and gathers the following information:

- All customers insure at least one car.

- 70 percent of the customers insure more than one car.
- 20 percent of the customers insure a sports car.
- Of those customers who insure more than one car, 15 percent insure a sports car.

(a) Calculate the probability that a randomly selected customer insures exactly one car and that car is not a sports car.
(b) Calculate the probability that a randomly selected customer insures more than one car, given that s/he insures a sports car.

6. Suppose that A, B, and C are events in a nonempty sample space S. Prove each of the following facts:
(a) If $P(A|B) = P(A|\overline{B})$, then A and B are independent.
(b) If $P(A|C) > P(B|C)$ and $P(A|\overline{C}) > P(B|\overline{C})$, then $P(A) > P(B)$.
(c) $P(A \cup B \cup C) \leq P(A) + P(B) + P(C)$.

7. A transmitter is sending a message in binary code ("+" and "−" signals) that must pass through two independent relay stations before being sent on to the receiver. Schematically, the message is sent as follows

$$\text{Transmitter} \Longrightarrow \text{Relay 1} \Longrightarrow \text{Relay 2} \Longrightarrow \text{Receiver.}$$

At each relay station, there is a 25 percent chance that a signal will be reversed; that is, for $i = 1, 2$,

$$\begin{aligned} P(\text{``}+\text{'' is sent by relay } i | \text{``}-\text{'' is received by relay } i) &= 0.25 \\ P(\text{``}-\text{'' is sent by relay } i | \text{``}+\text{'' is received by relay } i) &= 0.25. \end{aligned}$$

Suppose that "+" symbols make up 60 percent of the messages being sent by the transmitter. If a "+" is received from Relay 2, what is the probability that a "+" was sent by the transmitter?

8. Suppose that n women at a party throw their hats in the center of the room. Each woman then randomly selects a hat. Denote by A the event that none of the women selects her own hat.
(a) Show that

$$P(A) = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \cdots \pm \frac{(-1)^n}{n!}.$$

(b) As n grows large without bound (i.e., as $n \to \infty$), what does $P(A)$ converge to? Does this answer agree with your intuition? Why or why not?

9. Let S be a sample space, and suppose that events A and B are subsets of S. Show that

$$P(A \cap B) \geq 1 - P(\overline{A}) - P(\overline{B}).$$

Your argument should be convincing and well-written. *Hint.* One way to prove this is to start with the Inclusion-Exclusion property.

10. Suppose that individuals in a large population are infected with a certain disease with probability p; that is, each individual in the population has the same probability of being positive. Assume that all individuals in the population are independent. As part of a disease-monitoring program, a sample of n individuals is chosen.
(a) Find an expression for the probability that exactly 1 individual is positive.
(b) Find an expression for the probability that at least 1 individual is positive.
(c) Find an expression for the probability that exactly 1 individual is positive, given that at least 1 individual is positive.
(d) With $n = 5$, view your answer to part (c) as a function of p, defined over $(0, 1)$, and graph it. Also with $n = 5$, find the smallest value of p for which your answer in part (c) is less than 0.5.

11. Suppose that S is a sample space and that A and B are subsets of S with $P(A) = 0.4$, $P(B) = 0.5$, and $P(A \cap B) = 0.3$. Compute each of the following probabilities:
(a) $P(\overline{A})$
(b) $P(A \cup B)$
(c) $P(\overline{A} \cap B)$
(d) $P(\overline{A} \cup \overline{B})$
(e) $P(B|A)$
(f) Are A and B independent? Why or why not?

12. A box contains $4N$ balls; $2N$ of the balls are white, $2N$ are black. The following trial is performed:

(i) $2N$ of the balls are selected at random (equally likely) and without replacement. If the event

$$A = \{\text{exactly } N \text{ of the selected balls are black}\}$$

occurs, then the experiment is stopped.

(ii) Otherwise, the $2N$ balls are replaced and the trial described in (i) is repeated.

(a) Find an expression for $P(A)$; your expression should depend on N.
(b) Find an expression for the probability that exactly 3 trials are required to observe A. Assume that the trials are independent.

13. I have 8 students: Jack, Jill, Fred, Fran, Clinton, Corrinne, Wes, and Wanda. I will choose 4 of these 8 students for a committee to study global warming. The committee posts are not distinct. Students are selected at random.
(a) Describe an appropriate sample space for this experiment. How many possible committees are there?
(b) Suppose that the committee chosen was Jack, Fred, Clinton, and Wes (all males; the remaining students are females). If the selection process was truly random, what is the probability of selecting this committee?
(c) How many possible committees are there if Jack and Jill will not serve together?
(d) How many possible committees are there if Clinton and Corrinne will serve together or not at all?

14. A life insurance company issues standard, preferred, and ultra-preferred policies. Of the company's policy holders of a certain age,

- 60 percent are standard with a probability of 0.05 of dying next year.
- 30 percent are preferred with a probability of 0.03 of dying next year.
- 10 percent are ultra-preferred with a probability of 0.01 of dying next year.

(a) What is the probability that a policy holder of this certain age dies next year?
(b) A policy holder of this certain age dies next year. What is the probability that the deceased was a preferred policy holder?
(c) A policy holder of this certain age does not die next year. What is the probability this policy holder was an ultra-preferred policy holder?

15. Suppose that S is a non-empty sample space, and let $P(\cdot)$ denote a probability measure on S. Using appropriate notation, give the three axioms that $P(\cdot)$ satisfies.

16. Suppose that S is a non-empty sample space and that A and B are subsets of S with $P(A) = 0.4$, $P(B) = 0.5$, and $P(A \cap B) = 0.3$. Compute each of the following probabilities:
(a) $P(\overline{A})$
(b) $P(A \cup B)$
(c) $P(\overline{A} \cap B)$
(d) $P(\overline{A} \cup \overline{B})$

17. In one class I taught recently, there were 6 undergraduate students and 26 graduates. To investigate the studying habits of students in this course, I decided to pick a committee of 3 students from the class (at random and without replacement) to help me collect information from all 32 students. The 3 committee posts were **not distinct**; that is, there are no designations of president, vice president, or anything like that.
(a) Suppose that I wanted one undergraduate and two graduates on the committee. How many different 1-undergrad, 2-grad-committees are possible?
(b) Suppose that I just randomly selected three students from the class (at random and without replacement). What is the probability that there will be **at least one** undergraduate on the committee?

18. Suppose that A and B are events in a nonempty sample space S.
(a) Show that if $P(A|B) = P(A|\overline{B})$, then A and B must be independent events. *Hint*: You might use the Law of Total Probability.
(b) Is the converse to part (a) always true? That is, if A and B are independent, does $P(A|B) = P(A|\overline{B})$ necessarily hold? Prove or give a counterexample.

19. A drawer contains 3 black, 5 green, and 2 red socks. Two socks are selected at random from the drawer. Assume that socks of the same color can not be distinguished from each other.

(a) Write out a sample space for this experiment. Are the sample points in your sample space equally likely? Explain.
(b) Compute the probability that both socks are the same color.

20. Suppose that S is a non-empty sample space, and let P denote a probability measure on S which satisfies our three **Kolmolgorov axioms**. For any event B with $P(B) > 0$, recall that we defined

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

Prove each of the following:

1. $P(A|B) \geq 0$
2. $P(B|B) = 1$
3. If $A_1, A_2, ...$ is a countable sequence of pairwise disjoint events (i.e., $A_i \cap A_j = \emptyset$, for $i \neq j$) in S, then
$$P\left(\bigcup_{i=1}^{\infty} A_i \middle| B\right) = \sum_{i=1}^{\infty} P(A_i|B).$$

21. A plane is missing and is presumed to have crashed in one of three regions: R_1, R_2, or R_3 (no other regions are possible). Field experts from the NTSB have projected that

- there is a 30 percent chance that the plane crashed in region 1
- there is a 20 percent chance that the plane crashed in region 2
- there is a 50 percent chance that the plane crashed in region 3.

A search party will be successful at finding the crashed plane with probability 0.8 for region 1, 0.9 for region 2, and 0.3 for region 3.
(a) What is the probability that the crashed plane will be found in region 2?
(b) What is the probability that the crashed plane will be found?
(c) What is the probability that the plane crashed in region 3 **given** that the search was successful?

22. In one class I taught recently, there were **4 undergraduates** and **18 graduates**. To investigate the studying habits of students in this course, I decided to pick a committee of 3 students from the class (at random and without replacement) to help me collect information from all 22 students. The 3 committee posts were **not distinct**; that is, there are no designations of president, vice president, or anything like that.
(a) Suppose that I wanted one undergraduate and two graduates on the committee. How many different 1-undergrad, 2-grad-committees are possible?
(b) Suppose that I just randomly selected three students from the class (at random and without replacement). What is the probability that there will be **at least one** undergraduate on the committee?

23. A track star runs two races on a certain day. Let A_1 be the event that she wins the first race and let A_2 be the event that she wins the second race. Suppose that $P(A_1) = 0.6, P(A_2) = 0.3$, and $P(A_1 \cap A_2) = 0.1$.
(a) Find the probability that she wins the second race given that she wins the first race.
(b) Compute the probability that she wins neither race.
(c) Are the events A_1 and A_2 independent? Justify your answer.

24. A room contains 9 students: 4 are chemistry majors, 3 are business majors, and 2 are psychology majors. All of the students are lined up, at random, in the room. What is the probability that the two psychology majors are lined up next to each other?

25. The use of plant appearance in prospecting for ore deposits is called *geobotanical prospecting*. One indicator of copper is a small mint with a mauve-colored flower. Suppose that, for a certain region, there is a 30 percent chance that the soil has a high copper content and a 23 percent chance that the mint will be present there. In addition, we know that if the copper content is high, there is a 70 percent chance that the mint will be present. Let C denote the event that a soil sample has high copper content, and let M denote the event that the mint is present.
(a) Find the probability that the copper content will be high **and** the mint will be present.
(b) Find the probability that the copper content will be high **given** that the mint is present.

26. A project manager has 10 chemical engineers on her staff. Four are women and six are men. The engineers are equally qualified, so the project manager randomly selects three engineers (order is **not** important). What is the probability that no women are selected? Given your calculation, would you consider it unusual if no women were selected?

27. In Kenya, the HIV prevalence in the adult population is about 14%. In a public-health screening study in Nairobi (the capital of Kenya), an enzyme-linked immunosorbant assay (ELISA) test kit is used to determine whether or not individuals have the HIV infection. The particular ELISA test used is known to give a false positive 0.5% of the time; that is, it will produce a positive result 0.5% of the time when the disease is **not** present. In addition, the ELISA test correctly gives a positive result when the disease is really present 99% of the time. In this study, suppose that a randomly selected adult individual tests positive for the disease. What is the probability that the person actually has the disease? Assume that the prevalence of HIV in Nairobi is approximately that of the national average.

28. State and prove Bayes Rule for two events A and B.

CHAPTER 3 PROBLEMS

29. I am preparing an itinerary to visit 5 cities: Birmingham, Raleigh, Iowa City, Seattle, and Dallas. The order in which I visit the cities will determine the cost of the entire trip.
(a) How many different itineraries are possible?
(b) Let Y denote the number of cities visited before Iowa City. Find the probability mass function of Y. Assume that each itinerary is **equally likely**.
(c) Use your pmf in part (b) to compute $E(Y)$.

30. Suppose that Y is a random variable with $E(Y) = 3$ and $V(Y) = 1$.
(a) Compute $E[(Y-1)^2]$.
(b) Compute $V(2-3Y)$.

31. A **truncated discrete distribution** arises when a particular value cannot be observed, and is eliminated from the support. In particular, if Y has the pmf $p_Y(y)$ with support $0, 1, 2, ...$, and the "0" value cannot be observed, the **0−truncated** random variable Y_T has pmf $p_{Y_T}(y)$, where

$$p_{Y_T}(y) = \begin{cases} p_Y(y)/[1-p_Y(0)], & y = 1, 2, 3, ..., \\ 0, & \text{otherwise.} \end{cases}$$

For $0 < \theta < 1$, suppose that the random variable Y has pmf

$$p_Y(y) = \begin{cases} \theta(1-\theta)^y, & y = 0, 1, 2, ..., \\ 0, & \text{otherwise,} \end{cases}$$

(a) Show that the pmf of the 0−truncated version of Y, Y_T, is given by

$$p_{Y_T}(y) = \begin{cases} \theta(1-\theta)^{y-1}, & y = 1, 2, 3, ..., \\ 0, & \text{otherwise.} \end{cases}$$

(b) Find the moment generating function of Y_T in part (a) and use it to compute $E(Y_T)$. Note that you can do this part even if you could not do part (a).

32. Suppose that Y is a discrete random variable with pmf given by

y	0	1	4	9
$p_Y(y)$	0.5	0.2	0.2	0.1

(a) Compute $E(Y)$ and $V(Y)$ without using the mgf of Y.
(b) Find the mgf of Y.
(c) Compute $E(Y)$ and $V(Y)$ using the mgf of Y.

33. An insurance company models the number of "serious injury" claims per day, Y, as a random variable with pmf

$$p_Y(y) = \begin{cases} \frac{1}{(y+1)(y+2)}, & y = 0, 1, 2, ... \\ 0, & \text{otherwise.} \end{cases}$$

(a) Show that this is a valid pmf; that is, show that

$$\sum_{y=0}^{\infty} p_Y(y) = 1.$$

Hint: It is helpful to write

$$\frac{1}{(y+1)(y+2)} = \frac{1}{y+1} - \frac{1}{y+2}$$

and then subsequently recognize $\sum_{y=0}^{\infty} p_Y(y)$ as a telescoping sum.
(b) Find the probability that on a given day, there are at least two serious injury claims.
(c) Find the probability that on a given day, there are exactly two serious injury claims, if it is known that there is at least one claim made (this is a conditional probability).
(d) Show analytically that $E(Y) = \infty$. *Hint:* Try to find a function $g(y)$ such that

$$\frac{y}{(y+1)(y+2)} > g(y),$$

for all $y \geq 1$, where the function $g(y)$ is not integrable over $[1, \infty)$; i.e., $\int_1^{\infty} g(y)dy = \infty$. The result then follows from the integral comparison test from calculus.
(e) If $p_Y(y)$ is the correct model for Y, and the company pays every claim, how long do you think this company will stay in business?

34. Two fair dice are rolled. Let Y equal the product of the 2 dice.
(a) Find the pmf of Y.
(b) Find the mean and variance of Y.

35. Let Y be a random variable with a discrete pmf

$$p_Y(y) = \begin{cases} y/8, & y = 1, 2, 5 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Graph the probability histogram for Y.
(b) Find $E(Y)$ and $V(Y)$ without using the mgf.
(c) Find $E(2Y + 3)$ and $V(2Y + 3)$.
(d) Find the mgf of Y.
(e) Find $E[Y^Y - Y \ln(3Y)]$.

36. Suppose that the random variable Y has pmf

$$p_Y(y) = \begin{cases} \theta(1-\theta)^y, & y = 0, 1, 2, ..., \\ 0, & \text{otherwise}, \end{cases}$$

where $0 < \theta < 1$.
(a) Show that $p_Y(y)$ is a valid pmf.
(b) Derive the mgf of Y, and use it to find expressions for $E(Y)$ and $V(Y)$.

37. Suppose that the random variable X has the pmf

$$p_X(x) = \begin{cases} \frac{1}{12}(c - 2x), & x = 0, 1, 2 \\ 0, & \text{otherwise}. \end{cases}$$

(a) Find the value of c which makes this a valid pmf.
(b) Compute $E(X)$.

38. Suppose that a and b are real constants. For any random variable X, show that

$$V(a + bX) = b^2 V(X).$$

39. Suppose that the random variable Y has moment generating function $m_Y(t)$. Let $r_Y(t) = \ln m_Y(t)$. Show that

$$V(Y) = \frac{d^2}{dt^2}\, r_Y(t)\bigg|_{t=0}.$$

The function $r_Y(t)$ is called the **cumulant generating function**.
Hint. Recall from calculus that if $f(t)$ is a function of t, then $\frac{d}{dt}[\ln f(t)] = f'(t)/f(t)$, where $f'(t)$ denotes the derivative of $f(t)$ with respect to t.

40. Suppose that the random variable X has probability mass function (pmf)

$$p_X(x) = \begin{cases} ce^{-x}, & x = 1, 2, ... \\ 0, & \text{otherwise}, \end{cases}$$

where the constant $c = e/(e-1)$.
(a) Show that this pmf is **valid**; i.e., show that $\sum_{x=1}^{\infty} p_X(x) = 1$.
(b) Compute $P(X \geq 3)$.
(c) Show that the moment generating function of X is given by

$$M_X(t) = c(1 - e^{t-1})^{-1},$$

for values of $t < 1$. Make sure you argue why $t < 1$.

41. I have two boxes. Each box has four balls in it; numbered 1, 2, 3, and 4, respectively. I am going to conduct the following experiment: I randomly select two balls, one from the first box and one from the second box, and record the number on each ball. Define the random variables

$$\begin{aligned} X &= \text{minimum ball value,} \\ Y &= \text{maximum ball value, and} \\ R &= Y - X. \end{aligned}$$

For example, if I selected a 2 and a 3, then $y = 3$, $x = 2$, and $r = y - x = 3 - 2 = 1$. Or, if I selected a 4 and a 4, then $y = 4$, $x = 4$, and $r = y - x = 4 - 4 = 0$.
(a) Find the pdf of R, and sketch a graph of it. Label everything on your graph.
(b) Compute $E(R)$ and $V(R)$.

42. A discrete random variable Y has pmf

$$p_Y(y) = \begin{cases} c(1/2)^y, & y = 1, 2, 3, ... \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the value of c.
(b) Compute $P(Y \leq 2)$.
(c) Derive the mgf of Y.
(d) Find $E(Y)$ and $V(Y)$.

43. A group contains 12 persons. **Three** of these people will be chosen, at random and without replacement, to participate in a public opinion survey. For the following questions, denote by R, D, and I, respectively, a republican, a democrat, and an independent. In what follows, ordering here is not important because each chosen person will be given the same survey.
(a) If the group contains 4R, 5D, and 3I, what is the probability that the committee will contain exactly one member from each party (i.e., 1R, 1D, 1I).
(b) If the group contains 6R, 4D, and 2I, what is the probability that the committee will contain individuals who all belong to the same party?
(c) If the group contains 5R, 6D, and 1I, what is the probability that there will be no democrats chosen?
(d) In part (c), let Y denote the number of independents chosen. Find the pmf for Y and graph it using a probability histogram (note that Y is either 0 or 1).

44. Suppose that Y is a random variable with $E(Y) = 1$ and $V(Y) = 5$. Find
(a) $E[(Y + 2)^2]$.
(b) $V(4 + 3Y)$.

45. Let Y be a random variable with $P(Y = 1) = p = 1 - P(Y = 0)$.
(a) Find the mean and variance of Y.
(b) For what value of p is $V(Y)$ minimized?
(c) Find $c \neq 1$ such that $E(c^Y) - 1 = 0$.