Review of continuous random variables

I: General continuous distributions. A random variable Y is said to be a continue random variable if the cumulative distribution function (cdf), denoted by $F_Y(y)$, of Y is continuous.

- $F_Y(y) = P(Y \le y)$ for both discrete and continuous random variables.
- $F_Y(y)$ is defined on the whole real line; i.e., for $-\infty < y < +\infty$.
- See lecture notes Page 64 for the three properties of a cumulative distribution function.
- You should know how to calculate a cdf from a pmf (see Example 4.1 on page 62 of the notes)
- Also should know how to calculate a cdf from a pdf (see Example 4.3 on page 67 of the notes)
- For a continuous random variable Y, its probability density function (pdf), denoted by $f_Y(y)$, is defined by

  $$f_Y(y) = \frac{d}{dy} F_Y(y).$$

  This is how to calculate $f_Y(y)$ from $F_Y(y)$.

- Thus to find $F_Y(y)$ from $f_Y(y)$ is simply

  $$F_Y(y) = \int_{-\infty}^{y} f_Y(u)du.$$

- To check whether a function is a valid pdf, two things need be checked:

  $$(1)\ f_Y(y) \ge 0; \quad (2) \int_{-\infty}^{\infty} f_Y(y)dy = 1.$$

Two ways to calculate probability, based on cdf or pdf:

- $P(Y \le b) = F_Y(b) = \int_{-\infty}^{b} f_Y(y)dy$.
- $P(Y \ge b) = 1 - F_Y(b) = \int_{b}^{+\infty} f_Y(y)dy$.
- $P(a \le Y \le b) = F_Y(b) - F_Y(a) = \int_{a}^{b} f_Y(y)dy$.

Important quantities of Y:

- The pth quantile of the distribution of Y, denoted by ϕ_p solves

$$F_Y(\phi_p) = P(Y \le \phi_p) = p.$$

- Expectation

$$E(Y) = \int_{-\infty}^{\infty} y f_Y(y) dy.$$

  Note $E(aY + b) = aE(Y) + b$.

- Variance

$$V(Y) = E(Y^2) - [E(Y)]^2$$

  where

$$E(Y^2) = \int_{-\infty}^{\infty} y^2 f_Y(y) dy.$$

  Note $V(aY + b) = a^2 V(Y)$.

- Actually

$$E(Y^k) = \int_{-\infty}^{\infty} y^k f_Y(y) dy.$$

  You can certainly use the moment generating function to compute $E(Y^k)$, but sometimes, this integral is much easier than taking derivatives of mgf.

- Moment generating function

$$m_Y(t) = E(e^{tY}) = \int_{-\infty}^{\infty} e^{ty} f_Y(y) dy,$$

  defined on the collection of t that makes $E(e^{tY})$ finite. The main use of $m_Y(t)$ is to generate moments:

$$E(Y^k) = \frac{d^k}{dt^k} m_Y(t) \bigg|_{t=0}.$$

II: Uniform distributions. $Y \sim \mathcal{U}(\theta_1, \theta_2)$, where $\theta_1 < \theta_2$. The support of Y is $\{y : \theta_1 < y < \theta_2\}$.

1. pdf

$$f_Y(y) = \begin{cases} \frac{1}{\theta_2 - \theta_1}, & \theta_1 < y < \theta_2, \\ 0, & \text{otherwise.} \end{cases}$$

2. cdf

$$F_Y(y) = \begin{cases} 0, & y \leq \theta_1, \\ \frac{y - \theta_1}{\theta_2 - \theta_1}, & \theta_1 < y < \theta_2, \\ 1, & y \geq \theta_2. \end{cases}$$

3. $E(Y) = \frac{\theta_1 + \theta_2}{2}$, $V(Y) = \frac{(\theta_2 - \theta_1)^2}{12}$

4. mgf:

$$m_Y(t) = \begin{cases} \frac{\exp(\theta_2 t) - \exp(\theta_1 t)}{t(\theta_2 - \theta_1)}, & t \neq 0, \\ 1, & t = 0. \end{cases}$$

5. Find probability of type $P(Y > a)$, $P(Y < b)$, $P(a < Y < b)$.

III: Normal distributions. $Y \sim \mathcal{N}(\mu, \sigma^2)$, where $\sigma > 0$. The support of Y is $\{y : -\infty < y < \infty\}$.

1. pdf

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{ -\frac{1}{2} \left(\frac{y - \mu}{\sigma} \right)^2 \right\}.$$

   Note that it is **symmetric** with respect to $y = \mu$.

2. $E(Y) = \mu$, $V(Y) = \sigma^2$

3. mgf: $m_Y(t) = \exp(\mu t + \sigma^2 t^2 / 2)$.

4. Standardization:

$$Z = \frac{Y - \mu}{\sigma} \sim \mathcal{N}(0, 1).$$

5. $P(a < Y < b) = \mathit{normalcdf}(a, b, \mu, \sigma)$, where a, b could be $\pm\infty$, then put $\pm 10^{99}$.

6. For $Y \sim \mathcal{N}(\mu, \sigma^2)$, the pth quantile $\phi_p = \mathit{invNorm}(p, \mu, \sigma)$.

IV: Exponential distributions. $Y \sim \text{exponential}(\beta)$, where $\beta > 0$. The support is $\{y : y > 0\}$.

1. pdf

$$f_Y(y) = \begin{cases} \frac{1}{\beta}\exp(-y/\beta), & y > 0, \\ 0, & \text{otherwise.} \end{cases}$$

2. cdf

$$F_Y(y) = \begin{cases} 0, & y \leq 0, \\ 1 - \exp(-y/\beta), & y > 0. \end{cases}$$

3. $E(Y) = \beta$, $V(Y) = \beta^2$

4. mgf:

$$m_Y(t) = (1 - \beta t)^{-1}, \quad \text{for } t < \beta^{-1}.$$

5. Memoryless property: for any $r > 0$ and $s > 0$,

$$P(Y > r + s | Y > r) = P(Y > s).$$

6. Relationship with a Poisson process. Suppose that we are observing events according to a Poisson process with rate $\lambda = \beta^{-1}$, and let the random variable Y denote the time until the first occurrence. Then $Y \sim \text{exponential}(\beta)$

7. Find probability of type $P(Y > a)$, $P(Y < b)$, $P(a < Y < b)$.

V: Gamma distributions. $Y \sim \text{gamma}(\alpha, \beta)$, where $\alpha > 0$, $\beta > 0$. The support of Y is $\{y : y > 0\}$.

1. Gamma function

$$\Gamma(t) = \int_0^\infty y^{t-1}\exp(-y)dy, \quad \text{for} t > 0.$$

for a positive integer n, $\Gamma(n) = (n-1)!$.

2. An important formula:

$$\Gamma(\alpha)\beta^\alpha = \int_0^\infty y^{\alpha-1}\exp(-y/\beta)dy.$$

3. pdf

$$f_Y(y) = \begin{cases} \frac{1}{\Gamma(\alpha)\beta^\alpha} y^{\alpha-1} \exp(-y/\beta), & y > 0, \\ 0, & \text{otherwise.} \end{cases}$$

4. $E(Y) = \alpha\beta$, $V(Y) = \alpha\beta^2$

5. mgf:

$$m_Y(t) = (1 - \beta t)^{-\alpha}, \quad \text{for } t < \beta^{-1}.$$

6. Relationship with a Poisson process. Suppose that we are observing events according to a Poisson process with rate $\lambda = \beta^{-1}$, and let the random variable Y denote the time until the αth occurrence. Then $Y \sim \text{gamma}(\alpha, \beta)$

7. Find probability of type $P(Y > a)$, $P(Y < b)$, $P(a < Y < b)$. TI-84 for definite integrals.

VI: χ^2 distributions. $Y \sim \chi^2(\nu)$ with ν being a positive integer, a special case of Gamma distributions when $\alpha = \nu/2$ and $\beta = 2$. The support of Y is $\{y : y > 0\}$.

1. pdf

$$f_Y(y) = \begin{cases} \frac{1}{\Gamma(\nu/2)2^\nu} y^{(\nu/2)-1} \exp(-y/2), & y > 0, \\ 0, & \text{otherwise.} \end{cases}$$

2. $E(Y) = \nu$, $V(Y) = 2\nu$

3. mgf:

$$m_Y(t) = (1 - 2t)^{-\nu/2}, \quad \text{for } t < 2^{-1}.$$

4. Find the upper αth quantile χ^2_α using the Table.

5. Find probability of type $P(Y > a)$, $P(Y < b)$, $P(a < Y < b)$. TI-84 for definite integrals.

VII: Beta distributions. $Y \sim \text{beta}(\alpha, \beta)$, $\alpha > 0, \beta > 0$. The support of Y is $\{y : 0 < y < 1\}$.

1. An important formulation

$$\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} = \int_0^1 y^{\alpha-1}(1-y)^{\beta-1}dy.$$

2. pdf

$$f_Y(y) = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} y^{\alpha-1}(1-y)^{\beta-1}, & 0 < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

3. $E(Y) = \alpha/(\alpha+\beta)$, $V(Y) = (\alpha\beta)/\{(\alpha+\beta)^2(\alpha+\beta+1)\}$

4. mgf: no closed form

5. Find probability of type $P(Y > a)$, $P(Y < b)$, $P(a < Y < b)$. TI-84 for definite integrals.

6. Solve for quantiles, especially when $\alpha = 1$, or $\beta = 1$, or both α and β are small integers (check Example 4.17 on page 93).