Notes_2_6_Tools_for_Counting_Sample_Points

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Example 2.9. Two jurors are needed from a pool of 2 men and 2 women. The jurors are randomly selected from the 4 individuals. A sample space for this experiment is

$$S = \{(M1, M2), (M1, W1), (M1, W2), (M2, W1), (M2, W2), (W1, W2)\}.$$

What is the probability that the two jurors chosen consist of 1 male and 1 female?
SOLUTION. There are $N = 6$ sample points, denoted in order by $E_1, E_2, ..., E_6$. Let the event

$$A = \{\text{one male, one female}\} = \{(M1, W1), (M1, W2), (M2, W1), (M2, W2)\},$$

so that $n_A = 4$. If the sample points are equally likely (probably true if the jurors are randomly selected), then $P(A) = 4/6$. $\square$

2.6 Tools for counting sample points

2.6.1 The multiplication rule

MULTIPLICATION RULE: Consider an experiment consisting of $k \geq 2$ "stages," where

$$\begin{aligned} n_1 &= \text{number of ways stage 1 can occur} \\ n_2 &= \text{number of ways stage 2 can occur} \\ &\vdots \\ n_k &= \text{number of ways stage } k \text{ can occur.} \end{aligned}$$

Then, there are

$$\prod_{i=1}^{k} n_i = n_1 \times n_2 \times \cdots \times n_k$$

different outcomes in the experiment.

Example 2.10. An experiment consists of rolling two dice. Envision stage 1 as rolling the first and stage 2 as rolling the second. Here, $n_1 = 6$ and $n_2 = 6$. By the multiplication rule, there are $n_1 \times n_2 = 6 \times 6 = 36$ different outcomes. $\square$

Example 2.11. In a controlled field experiment, I want to form all possible treatment combinations among the three factors:

Factor 1: Fertilizer (60 kg, 80 kg, 100kg: 3 levels)

Factor 2: Insects (infected/not infected: 2 levels)

Factor 3: Precipitation level (low, high: 2 levels).

Here, $n_1 = 3, n_2 = 2$, and $n_3 = 2$. Thus, by the multiplication rule, there are $n_1 \times n_2 \times n_3 = 12$ different treatment combinations. □

Example 2.12. Suppose that an Iowa license plate consists of seven places; the first three are occupied by letters; the remaining four with numbers. Compute the total number of possible orderings if

(a) there are no letter/number restrictions.

(b) repetition of letters is prohibited.

(c) repetition of numbers is prohibited.

(d) repetitions of numbers and letters are prohibited.

ANSWERS:

(a) $26 \times 26 \times 26 \times 10 \times 10 \times 10 \times 10 = 175,760,000$

(b) $26 \times 25 \times 24 \times 10 \times 10 \times 10 \times 10 = 156,000,000$

(c) $26 \times 26 \times 26 \times 10 \times 9 \times 8 \times 7 = 88,583,040$

(d) $26 \times 25 \times 24 \times 10 \times 9 \times 8 \times 7 = 78,624,000$

2.6.2 Permutations

TERMINOLOGY: A **permutation** is an arrangement of distinct objects in a particular order. *Order is important.*

PROBLEM: Suppose that we have n distinct objects and we want to **order** (or **permute**) these objects. Thinking of n slots, we will put one object in each slot. There are

- n different ways to choose the object for slot 1,
- $n-1$ different ways to choose the object for slot 2,
- $n-2$ different ways to choose the object for slot 3,

and so on, down to

- 2 different ways to choose the object for slot $(n-1)$, and
- 1 way to choose for the last slot.

IMPLICATION: By the multiplication rule, there are $n(n-1)(n-2)\cdots(2)(1) = n!$ different ways to order (permute) the n distinct objects.

Example 2.13. My bookshelf has 10 books on it. How many ways can I permute the 10 books on the shelf? ANSWER: $10! = 3,628,800$. □

Example 2.14. Now, suppose that in Example 2.13 there are 4 math books, 2 chemistry books, 3 physics books, and 1 statistics book. I want to order the 10 books so that all books of the same subject are together. How many ways can I do this?
SOLUTION: Use the multiplication rule.

Stage 1	Permute the 4 math books	4!
Stage 2	Permute the 2 chemistry books	2!
Stage 3	Permute the 3 physics books	3!
Stage 4	Permute the 1 statistics book	1!
Stage 5	Permute the 4 subjects $\{m, c, p, s\}$	4!

Thus, there are $4! \times 2! \times 3! \times 1! \times 4! = 6912$ different orderings. □

PERMUTATIONS: With a collection of n distinct objects, we now want to choose and **permute** r of them ($r \leq n$). The number of ways to do this is

$$P_{n,r} \equiv \frac{n!}{(n-r)!}.$$

The symbol $P_{n,r}$ is read "the permutation of n things taken r at a time."
Proof. Envision r slots. There are n ways to fill the first slot, $n-1$ ways to fill the second slot, and so on, until we get to the rth slot, in which case there are $n-r+1$ ways to fill it. Thus, by the multiplication rule, there are

$$n(n-1)\cdots(n-r+1) = \frac{n!}{(n-r)!}$$

different permutations. □

Example 2.15. With a group of 5 people, I want to choose a committee with three members: a president, a vice-president, and a secretary. There are

$$P_{5,3} = \frac{5!}{(5-3)!} = \frac{120}{2} = 60$$

different committees possible. **Here, note that order is important.** □

Example 2.16. What happens if the objects to permute are **not distinct**? Consider the word $PEPPER$. How many permutations of the letters are possible?
TRICK: Initially, treat all letters as distinct objects by writing, say,

$$P_1E_1P_2P_3E_2R.$$

There are $6! = 720$ different orderings of these distinct objects. Now, there are

3! ways to permute the Ps
2! ways to permute the Es
1! ways to permute the Rs.

So, 6! is actually $3! \times 2! \times 1!$ times too large. That is, there are

$$\frac{6!}{3!\,2!\,1!} = 60 \text{ possible permutations. } \square$$

MULTINOMIAL COEFFICIENTS: Suppose that in a set of n objects, there are n_1 that are similar, n_2 that are similar, ..., n_k that are similar, where $n_1 + n_2 + \cdots + n_k = n$. The number of permutations (i.e., distinguishable permutations) of the n objects is given by the **multinomial coefficient**

$$\binom{n}{n_1 n_2 \cdots n_k} \equiv \frac{n!}{n_1!\ n_2!\ \cdots\ n_k!}.$$

NOTE: Multinomial coefficients arise in the algebraic expansion of the multinomial expression $(x_1 + x_2 + \cdots + x_k)^n$; i.e.,

$$(x_1 + x_2 + \cdots + x_k)^n = \sum_D \binom{n}{n_1 n_2 \cdots n_k} x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k},$$

where

$$D = \left\{ (n_1, n_2, ..., n_k) : \sum_{j=1}^{k} n_i = n \right\}.$$

Example 2.17. How many signals, each consisting of 9 flags in a line, can be made from 4 white flags, 2 blue flags, and 3 yellow flags?

ANSWER:

$$\frac{9!}{4!\ 2!\ 3!} = 1260. \ \square$$

Example 2.18. In Example 2.17, assuming all permutations are equally likely, what is the probability that all of the white flags are grouped together? We offer two solutions. The solutions differ in the way we construct the sample space. Define

$$A = \{\text{all four white flags are grouped together}\}.$$

SOLUTION 1. Work with a sample space that does **not** treat the flags as distinct objects, but merely considers color. Then, we know from Example 2.17 that there are 1260 different orderings. Thus,

$$N = \text{number of sample points in } S = 1260.$$

Let n_a denote the number of ways that A can occur. We find n_a by using the multiplication rule.

Stage 1	Pick four adjacent slots	$n_1 = 6$
Stage 2	With the remaining 5 slots, permute the 2 blues and 3 yellows	$n_2 = \frac{5!}{2!3!} = 10$

Thus, $n_a = 6 \times 10 = 60$. Finally, since we have equally likely outcomes, $P(A) = n_a/N = 60/1260 \approx 0.0476$. □

SOLUTION 2. Initially, treat all 9 flags as **distinct objects**; i.e.,

$$W_1W_2W_3W_4B_1B_2Y_1Y_2Y_3,$$

and consider the sample space consisting of the 9! different permutations of these 9 distinct objects. Then,

$$N = \text{number of sample points in } S = 9!$$

Let n_a denote the number of ways that A can occur. We find n_a, again, by using the multiplication rule.

Stage 1	Pick adjacent slots for W_1, W_2, W_3, W_4	$n_1 = 6$
Stage 2	With the four chosen slots, permute W_1, W_2, W_3, W_4	$n_2 = 4!$
Stage 3	With remaining 5 slots, permute B_1, B_2, Y_1, Y_2, Y_3	$n_3 = 5!$

Thus, $n_a = 6 \times 4! \times 5! = 17280$. Finally, since we have equally likely outcomes, $P(A) = n_a/N = 17280/9! \approx 0.0476$. □

$$\frac{6\times4!\times5!}{9!} = \frac{6\times\frac{5!}{2!3!}}{\frac{9!}{2!3!4!}} = \frac{6\times5!\times4!}{9!}$$

2.6.3 Combinations

COMBINATIONS: Given n distinct objects, the number of ways to choose r of them ($r \leq n$), *without regard to order*, is given by

$$C_{n,r} = \binom{n}{r} \equiv \frac{n!}{r!\,(n-r)!}.$$

The symbol $C_{n,r}$ is read "the combination of n things taken r at a time." By convention, we take $0! = 1$.

Proof: Choosing r objects is equivalent to breaking the n objects into two distinguishable groups:

Group 1	r chosen
Group 2	$(n-r)$ not chosen.

There are $C_{n,r} = \frac{n!}{r!(n-r)!}$ ways to do this. □

REMARK: We will adopt the notation $\binom{n}{r}$, read "n choose r," as the symbol for $C_{n,r}$. The terms $\binom{n}{r}$ are called **binomial coefficients** since they arise in the algebraic expansion of a binomial; viz.,

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r.$$

Example 2.19. Return to Example 2.15. Now, suppose that we only want to choose 3 committee members from 5 (without designations for president, vice-president, and secretary). Then, there are

$$\binom{5}{3} = \frac{5!}{3!\,(5-3)!} = \frac{5 \times 4 \times 3!}{3! \times 2!} = 10$$

different committees. □

NOTE: From Examples 2.15 and 2.19, one should note that

$$P_{n,r} = r! \times C_{n,r}.$$

Recall that combinations do not regard order as important. Thus, once we have chosen our r objects (there are $C_{n,r}$ ways to do this), there are then $r!$ ways to permute those r chosen objects. Thus, we can think of a permutation as simply a combination times the number of ways to permute the r chosen objects.

Example 2.20. A company receives 20 hard drives. Five of the drives will be randomly selected and tested. If all five are satisfactory, the entire lot will be accepted. Otherwise, the entire lot is rejected. If there are really 3 defectives in the lot, what is the probability of accepting the lot?

SOLUTION: First, the number of sample points in S is given by

$$N = \binom{20}{5} = \frac{20!}{5!\,(20-5)!} = 15504.$$

Negative Binomial Dist. (Later)

Let A denote the event that the lot is accepted. How many ways can A occur? Use the multiplication rule.

Stage 1	Choose 5 good drives from 17	$\binom{17}{5}$
Stage 2	Choose 0 bad drives from 3	$\binom{3}{0}$

By the multiplication rule, there are $n_a = \binom{17}{5} \times \binom{3}{0} = 6188$ different ways A can occur. Assuming an equiprobability model (i.e., each outcome is equally likely), $P(A) = n_a/N = 6188/15504 \approx 0.399$. □

2.7 Conditional probability

MOTIVATION: In some problems, we may be fortunate enough to have prior knowledge about the likelihood of events related to the event of interest. We may want to incorporate this information into a probability calculation.

TERMINOLOGY: Let A and B be events in a nonempty sample space S. The **conditional probability** of A, given that B has occurred, is given by

$$P(A|B) = \frac{P(A \cap B)}{P(B)},$$

provided that $P(B) > 0$.

Example 2.21. A couple has two children.

(a) What is the probability that both are girls?

(b) What is the probability that both are girls, if the eldest is a girl?