Notes_2_7_Conditional_Probability

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Notes_2_7
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SOLUTION: First, the number of sample points in S is given by

$$N = \binom{20}{5} = \frac{20!}{5!\,(20-5)!} = 15504.$$

Let A denote the event that the lot is accepted. How many ways can A occur? Use the multiplication rule.

Stage 1	Choose 5 good drives from 17	$\binom{17}{5}$
Stage 2	Choose 0 bad drives from 3	$\binom{3}{0}$

By the multiplication rule, there are $n_a = \binom{17}{5} \times \binom{3}{0} = 6188$ different ways A can occur. Assuming an equiprobability model (i.e., each outcome is equally likely), $P(A) = n_a/N = 6188/15504 \approx 0.399$. □

2.7 Conditional probability

MOTIVATION: In some problems, we may be fortunate enough to have prior knowledge about the likelihood of events related to the event of interest. We may want to incorporate this information into a probability calculation.

TERMINOLOGY: Let A and B be events in a nonempty sample space S. The **conditional probability** of A, given that B has occurred, is given by

$$P(A|B) = \frac{P(A \cap B)}{P(B)},$$

provided that $P(B) > 0$.

Example 2.21. A couple has two children.

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(a) What is the probability that both are girls? $.5 \times .5 = \frac{1}{4}$

(b) What is the probability that both are girls, if the eldest is a girl?

SOLUTION: (a) The sample space is given by

$$S = \{(M, M), (M, F), (F, M), (F, F)\}$$

and $N = 4$, the number of sample points in S. Define

$$A_1 = \{\text{1st born child is a girl}\},$$
$$A_2 = \{\text{2nd born child is a girl}\}.$$

Clearly, $A_1 \cap A_2 = \{(F, F)\}$ and $P(A_1 \cap A_2) = 1/4$, assuming that the four outcomes in S are equally likely.

SOLUTION: (b) Now, we want $P(A_2|A_1)$. Applying the definition of conditional probability, we get

$$P(A_2|A_1) = \frac{P(A_1 \cap A_2)}{P(A_1)} = \frac{1/4}{2/4} = 1/2. \square$$

Example 2.22. In a certain community, 36 percent of the families own a dog, 22 percent of the families that own a dog also own a cat, and 30 percent of the families own a cat. A family is selected at random.

(a) Compute the probability that the family owns both a cat and dog.

(b) Compute the probability that the family owns a dog, given that it owns a cat.

SOLUTION: Let $C = \{\text{family owns a cat}\}$ and $D = \{\text{family owns a dog}\}$. From the

SOLUTION: Let C = {family owns a cat} and D = {family owns a dog}. From the problem, we are given that $P(D) = 0.36$, $P(C|D) = 0.22$ and $P(C) = 0.30$. In (a), we want $P(C \cap D)$. We have

$$0.22 = P(C|D) = \frac{P(C \cap D)}{P(D)} = \frac{P(C \cap D)}{0.36}.$$

Thus,

$$P(C \cap D) = 0.36 \times 0.22 = 0.0792.$$

For (b), we want $P(D|C)$. Simply use the definition of conditional probability:

$$P(D|C) = \frac{P(C \cap D)}{P(C)} = \frac{0.0792}{0.30} = 0.264. \square$$

RESULTS: It is interesting to note that conditional probability $P(\cdot|B)$ satisfies the axioms for a probability set function when $P(B) > 0$. In particular,

1. $P(A|B) \geq 0$
2. $P(B|B) = 1$
3. If $A_1, A_2, ...$ is a countable sequence of **pairwise mutually exclusive** events (i.e., $A_i \cap A_j = \emptyset$, for $i \neq j$) in S, then

$$P\left(\bigcup_{i=1}^{\infty} A_i \middle| B\right) = \sum_{i=1}^{\infty} P(A_i|B).$$

EXERCISE. Show that the measure $P(\cdot|B)$ satisfies the Kolmolgorov axioms when $P(B) > 0$; i.e., establish the results above.

MULTIPLICATION LAW OF PROBABILITY: Suppose A and B are events in a non-empty sample space S. Then,

$$\begin{aligned} P(A \cap B) &= P(B|A)P(A) \\ &= P(A|B)P(B). \end{aligned}$$

$$
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P(A \cap B) &= P(B|A)P(A) \\
&= P(A|B)P(B).
\end{aligned}
$$

Proof. As long as $P(A)$ and $P(B)$ are strictly positive, this follows directly from the definition of conditional probability. □

EXTENSION: The multiplication law of probability can be extended to more than 2 events. For example,

$$
\begin{aligned}
P(A_1 \cap A_2 \cap A_3) &= P[(A_1 \cap A_2) \cap A_3] \\
&= P(A_3|A_1 \cap A_2) \times P(A_1 \cap A_2) \\
&= P(A_3|A_1 \cap A_2) \times P(A_2|A_1) \times P(A_1).
\end{aligned}
$$

NOTE: This suggests that we can compute probabilities like $P(A_1 \cap A_2 \cap A_3)$ "sequentially" by first computing $P(A_1)$, then $P(A_2|A_1)$, then $P(A_3|A_1 \cap A_2)$. The probability of a k-fold intersection can be computed similarly; i.e.,

$$
P\left(\bigcap_{i=1}^{k} A_i\right) = P(A_1) \times P(A_2|A_1) \times P(A_3|A_1 \cap A_2) \times \cdots \times P\left(A_k \middle| \bigcap_{i=1}^{k-1} A_i\right).
$$

Example 2.23. I am dealt a hand of 5 cards. What is the probability that they are all spades?

SOLUTION. Define A_i to be the event that card i is a spade ($i = 1, 2, 3, 4, 5$). Then,

$$
\begin{aligned}
P(A_1) &= \frac{13}{52} \\
P(A_2|A_1) &= \frac{12}{51} \\
P(A_3|A_1 \cap A_2) &= \frac{11}{50} \\
P(A_4|A_1 \cap A_2 \cap A_3) &= \frac{10}{49} \\
P(A_5|A_1 \cap A_2 \cap A_3 \cap A_4) &= \frac{9}{48},
\end{aligned}
$$

so that

13 12 11 10 9

so that

$$P(A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5) = \frac{13}{52} \times \frac{12}{51} \times \frac{11}{50} \times \frac{10}{49} \times \frac{9}{48} \approx 0.0005.$$

NOTE: As another way to solve this problem, a student recently pointed out that we could simply regard the cards as belonging to two groups: spades and non-spades. There are $\binom{13}{5}$ ways to draw 5 spades from 13. There are $\binom{52}{5}$ possible hands. Thus, the probability of drawing 5 spades (assuming that each hand is equally likely) is $\binom{13}{5}/\binom{52}{5} \approx 0.0005$. □

2.8 Independence

TERMINOLOGY: When the occurrence or non-occurrence of A has no effect on whether or not B occurs, and vice versa, we say that the events A and B are **independent**. Mathematically, we define A and B to be independent iff

$$P(A \cap B) = P(A)P(B).$$

Otherwise, A and B are called **dependent** events. Note that if A and B are independent,

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A)$$

and

$$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{P(B)P(A)}{P(A)} = P(B).$$