Notes_2_7_Conditional_Probability

Tuesday, August 30, 2016 9:46 AM



CHAPTER 2

STAT/MATH 511, J. TEBBS

SOLUTION: First, the number of sample points in S is given by

$$N = \binom{20}{5} = \frac{20!}{5! (20-5)!} = 15504.$$

Let A denote the event that the lot is accepted. How many ways can A occur? Use the multiplication rule.

Stage 1Choose 5 good drives from 17 $\binom{17}{5}$ Stage 2Choose 0 bad drives from 3 $\binom{3}{0}$

By the multiplication rule, there are $n_a = {\binom{17}{5}} \times {\binom{3}{0}} = 6188$ different ways A can occur. Assuming an equiprobability model (i.e., each outcome is equally likely), $P(A) = n_a/N = 6188/15504 \approx 0.399$. \Box

2.7 Conditional probability

MOTIVATION: In some problems, we may be fortunate enough to have prior knowledge about the likelihood of events related to the event of interest. We may want to incorporate this information into a probability calculation.

TERMINOLOGY: Let A and B be events in a nonempty sample space S. The conditional probability of A, given that B has occurred, is given by

$$P(A|B) = \frac{P(A \cap B)}{P(B)},$$

provided that P(B) > 0.

Example 2.21. A couple has two children.

Example 2.21. A couple has two children.

(a) What is the probability that both are girls?

(b) What is the probability that both are girls, if the eldest is a girl?

PAGE 17

CHAPTER 2

STAT/MATH 511, J. TEBBS

SOLUTION: (a) The sample space is given by

$$S = \{(M, M), (M, F), (F, M), (F, F)\}$$

and N = 4, the number of sample points in S. Define

$$A_1 = \{ \text{1st born child is a girl} \},\$$

$$A_2 = \{ \text{2nd born child is a girl} \}.$$

Clearly, $A_1 \cap A_2 = \{(F, F)\}$ and $P(A_1 \cap A_2) = 1/4$, assuming that the four outcomes in S are equally likely.

SOLUTION: (b) Now, we want $P(A_2|A_1)$. Applying the definition of conditional probability, we get

$$P(A_2|A_1) = \frac{P(A_1 \cap A_2)}{P(A_1)} = \frac{1/4}{2/4} = 1/2.$$

Example 2.22. In a certain community, 36 percent of the families own a dog, 22 percent of the families that own a dog also own a cat, and 30 percent of the families own a cat. A family is selected at random.

- (a) Compute the probability that the family owns both a cat and dog.
- (b) Compute the probability that the family owns a dog, given that it owns a cat.

SOLUTION: Let $C = \{\text{family owns a cat}\}$ and $D = \{\text{family owns a dog}\}$. From the

SOLUTION: Let $C = \{\text{family owns a cat}\}\ \text{and}\ D = \{\text{family owns a dog}\}\$. From the problem, we are given that P(D) = 0.36, P(C|D) = 0.22 and P(C) = 0.30. In (a), we want $P(C \cap D)$. We have

$$0.22 = P(C|D) = \frac{P(C \cap D)}{P(D)} = \frac{P(C \cap D)}{0.36}.$$

Thus,

$$P(C \cap D) = 0.36 \times 0.22 = 0.0792.$$

For (b), we want P(D|C). Simply use the definition of conditional probability:

$$P(D|C) = \frac{P(C \cap D)}{P(C)} = \frac{0.0792}{0.30} = 0.264. \ \Box$$

PAGE 18

CHAPTER 2

STAT/MATH 511, J. TEBBS

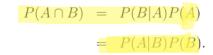
RESULTS: It is interesting to note that conditional probability $P(\cdot|B)$ satisfies the axioms for a probability set function when P(B) > 0. In particular,

- 1. $P(A|B) \ge 0$
- 2. P(B|B) = 1
- 3. If $A_1, A_2, ...$ is a countable sequence of **pairwise mutually exclusive** events (i.e., $A_i \cap A_j = \emptyset$, for $i \neq j$) in S, then

$$P\left(\bigcup_{i=1}^{\infty} A_i \middle| B\right) = \sum_{i=1}^{\infty} P(A_i | B).$$

EXERCISE. Show that the measure $P(\cdot|B)$ satisfies the Kolmolgorov axioms when P(B) > 0; i.e., establish the results above.

MULTIPLICATION LAW OF PROBABILITY: Suppose A and B are events in a nonempty sample space S. Then,



$$P(A \cap B) = P(B|A)P(A)$$
$$= P(A|B)P(B).$$

Proof. As long as P(A) and P(B) are strictly positive, this follows directly from the definition of conditional probability. \Box

EXTENSION: The multiplication law of probability can be extended to more than 2 events. For example,

$$P(A_1 \cap A_2 \cap A_3) = P[(A_1 \cap A_2) \cap A_3]$$

= $P(A_3 | A_1 \cap A_2) \times P(A_1 \cap A_2)$
= $P(A_3 | A_1 \cap A_2) \times P(A_2 | A_1) \times P(A_1).$

NOTE: This suggests that we can compute probabilities like $P(A_1 \cap A_2 \cap A_3)$ "sequentially" by first computing $P(A_1)$, then $P(A_2|A_1)$, then $P(A_3|A_1 \cap A_2)$. The probability of a k-fold intersection can be computed similarly; i.e.,

$$P\left(\bigcap_{i=1}^{k} A_i\right) = P(A_1) \times P(A_2|A_1) \times P(A_3|A_1 \cap A_2) \times \dots \times P\left(A_k \middle| \bigcap_{i=1}^{k-1} A_i \right).$$

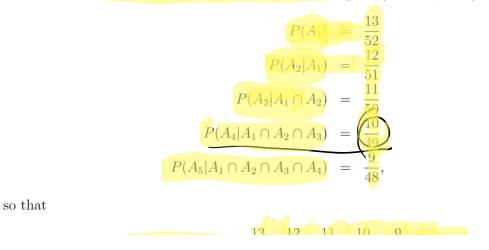
PAGE 19

CHAPTER 2

STAT/MATH 511, J. TEBBS

Example 2.23. I am dealt a hand of 5 cards. What is the probability that they are all spades?

SOLUTION. Define A_i to be the event that card *i* is a spade (i = 1, 2, 3, 4, 5). Then,



Quick Notes Page 4

so that

$$P(A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5) = \frac{13}{52} \times \frac{12}{51} \times \frac{11}{50} \times \frac{10}{49} \times \frac{9}{48} \approx 0.0005.$$

NOTE: As another way to solve this problem, a student recently pointed out that we could simply regard the cards as belonging to two groups: spades and non-spades. There are $\binom{13}{5}$ ways to draw 5 spades from 13. There are $\binom{52}{5}$ possible hands. Thus, the probability of drawing 5 spades (assuming that each hand is equally likely) is $\binom{13}{5}/\binom{52}{5} \approx 0.0005$. \Box

2.8 Independence

TERMINOLOGY: When the occurrence or non-occurrence of A has no effect on whether or not B occurs, and vice versa, we say that the events A and B are **independent**. Mathematically, we define A and B to be independent iff

$$P(A \cap B) = P(A)P(B).$$

Otherwise, A and B are called **dependent** events. Note that if A and B are independent,

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A)$$

and

$$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{P(B)P(A)}{P(A)} = P(B).$$

PAGE 20