Notes_2_8_Independence

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Notes_2_8_ Independ...

Example 2.23. I am dealt a hand of 5 cards. What is the probability that they are all spades?

SOLUTION. Define A_i to be the event that card i is a spade ($i = 1, 2, 3, 4, 5$). Then,

$$\begin{aligned}
P(A_1) &= \frac{13}{52} \\
P(A_2|A_1) &= \frac{12}{51} \\
P(A_3|A_1 \cap A_2) &= \frac{11}{50} \\
P(A_4|A_1 \cap A_2 \cap A_3) &= \frac{10}{49} \\
P(A_5|A_1 \cap A_2 \cap A_3 \cap A_4) &= \frac{9}{48},
\end{aligned}$$

so that

$$P(A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5) = \frac{13}{52} \times \frac{12}{51} \times \frac{11}{50} \times \frac{10}{49} \times \frac{9}{48} \approx 0.0005.$$

NOTE: As another way to solve this problem, a student recently pointed out that we could simply regard the cards as belonging to two groups: spades and non-spades. There are $\binom{13}{5}$ ways to draw 5 spades from 13. There are $\binom{52}{5}$ possible hands. Thus, the probability of drawing 5 spades (assuming that each hand is equally likely) is $\binom{13}{5}/\binom{52}{5} \approx 0.0005$. □

2.8 Independence

TERMINOLOGY: When the occurrence or non-occurrence of A has no effect on whether or not B occurs, and vice versa, we say that the events A and B are **independent**. Mathematically, we define A and B to be independent iff

$$P(A \cap B) = P(A)P(B).$$

Otherwise, A and B are called **dependent** events. Note that if A and B are independent,

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A)$$

and

$$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{P(B)P(A)}{P(A)} = P(B).$$

red white

$S = \{ (1,1) \ \vdots \ \}$ 36

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$S = \{(1,1), \ldots, (6,6)\}$ 36

$A = \{(4,1), (4,2), \ldots, (4,6)\}$

$B = \{(1,2), (1,4), (1,6), (2,1), (2,3), (2,5), \ldots, (6,1), (6,3), (6,5)\}$ 18

Example 2.24. A red die and a white die are rolled. Let $A = \{4 \text{ on red die}\}$ and $B = \{\text{sum is odd}\}$. Of the 36 outcomes in S, 6 are favorable to A, 18 are favorable to B, and 3 are favorable to $A \cap B$. Assuming the outcomes are equally likely,

$A \cap B = \{(4,1), (4,3), (4,5)\}$

$$\frac{1}{12} = \frac{3}{36} = P(A \cap B) = P(A)P(B) = \frac{6}{36} \times \frac{18}{36} = \frac{1}{6} \times \frac{1}{2} = \frac{1}{12}$$

numerically!!

and the events A and B are independent. $\square$

Example 2.25. In an engineering system, two components are placed in a **series**; that is, the system is functional as long as both components are. Let A_i; $i = 1, 2$, denote the event that component i is functional. Assuming independence, the probability the system is functional is then $P(A_1 \cap A_2) = P(A_1)P(A_2)$. If $P(A_i) = 0.95$, for example, then $P(A_1 \cap A_2) = 0.95 \times 0.95 = 0.9025$. If the events A_1 and A_2 are not independent, we do not have enough information to compute $P(A_1 \cap A_2)$. $\square$

INDEPENDENCE OF COMPLEMENTS: If A and B are independent events, so are

Demorgans law

(a) $\overline{A}$ and B ✓ $P(\overline{A} \cap B) = P(\overline{A}) \times P(B)$

(b) A and $\overline{B}$ ✓

(c) $\overline{A}$ and $\overline{B}$. $P(\overline{A} \cap \overline{B}) = P(\overline{A}) \times P(\overline{B})$?

$$\overline{A} \cap \overline{B} = \overline{A \cup B}$$

$$P(\overline{A} \cap \overline{B}) = P(\overline{A \cup B}) = 1 - P(A \cup B)$$
$$= 1 - (P(A) + P(B) - P(A \cap B))$$
$$= 1 - P(A) - P(B) + P(A)P(B)$$

Independence between A and B

$$= (1 - P(A))(1 - P(B))$$
$$= P(\overline{A})P(\overline{B})$$

Proof. We will only prove (a). The other parts follow similarly.

$$P(\overline{A} \cap B) = P(\overline{A}|B)P(B) = [1 - P(A|B)]P(B) = [1 - P(A)]P(B) = P(\overline{A})P(B). \quad \square$$

EXTENSION: The concept of independence (and independence of complements) can be extended to any finite number of events in S.

TERMINOLOGY: Let $A_1, A_2, ..., A_n$ denote a collection of $n \geq 2$ events in a nonempty sample space S. The events $A_1, A_2, ..., A_n$ are said to be **mutually independent** if for any subcollection of events, say, $A_{i_1}, A_{i_2}, ..., A_{i_k}$, $2 \leq k \leq n$, we have

$$P\left(\bigcap_{j=1}^{k} A_{i_j}\right) = \prod_{j=1}^{k} P(A_{i_j}).$$

HW

CHALLENGE: Come up with a random experiment and three events which are **pairwise independent**, but not mutually independent.

A_1, A_2, A_3 are pairwise Independent.

$P(A_1 \cap A_2) = P(A_1)P(A_2)$
$P(A_1 \cap A_3) = P(A_1)P(A_3)$
$P(A_2 \cap A_3) = P(A_2)P(A_3)$

Implies

A_1, A_2, A_3 are mutually Independent further requires $P(A_1 \cap A_2 \cap A_3)$

$= P(A_1)\,P(A_2)\,P(A_3)$

COMMON SETTING: Many experiments consist of a sequence of n trials that are viewed as independent (e.g., flipping a coin 10 times). If A_i denotes the event associated with the ith trial, and the trials are **independent**, then

$$P\left(\bigcap_{i=1}^{n} A_i\right) = \prod_{i=1}^{n} P(A_i).$$

Example 2.26. An unbiased die is rolled six times. Let $A_i = \{i \text{ appears on roll } i\}$, for $i = 1, 2, ..., 6$. Then, $P(A_i) = 1/6$, and assuming independence,

$$P(A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5 \cap A_6) = \prod_{i=1}^{6} P(A_i) = \left(\frac{1}{6}\right)^6.$$

Suppose that if A_i occurs, we will call it "a match." What is the probability of at least one match in the six rolls?

SOLUTION: Let B denote the event that there is at least one match. Then, $\overline{B}$ denotes the event that there are no matches. Now,

$$P(\overline{B}) = P(\overline{A}_1 \cap \overline{A}_2 \cap \overline{A}_3 \cap \overline{A}_4 \cap \overline{A}_5 \cap \overline{A}_6) = \prod_{i=1}^{6} P(\overline{A}_i) = \left(\frac{5}{6}\right)^6 = 0.335.$$

Thus, $P(B) = 1 - P(\overline{B}) = 1 - 0.335 = 0.665$, by the complement rule.

EXERCISE: Generalize this result to an n sided die. What does this probability converge to as $n \to \infty$? □

Let C denote the event there is exact 2 matches?

$P(C) = 6 \times \left(\frac{1}{6}\right) \times \left(\frac{5}{6}\right)^5$?

$\frac{1}{6}$ ✓ × × × × × $\left(\frac{5}{6}\right)^5$

$\frac{1}{6} \times \left(\frac{5}{6}\right)^5$ × ✓ × × × ×

$15 \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^4$

2.9 Law of Total Probability and Bayes Rule

SETTING: Suppose A and B are events in a nonempty sample space S. We can express the event A as follows

$$A = \underbrace{(A \cap B) \cup (A \cap \overline{B})}_{\text{union of disjoint events}}.$$

n-sided die   B: at least one matches

$P(B) = 1 - P(\overline{B}) = 1 - P(\text{no matches})$

$= 1 - P(\overline{A}_1 \cap \overline{A}_2 \cap \overline{A}_3 \cap \cdots \cap \overline{A}_n)$

$= 1 - \prod_{i=1}^{n} P(\overline{A}_i)$ $P(A_i) = \frac{1}{n}$

$$= 1 - \prod_{i=1}^{n} P(\bar{A}_i) \qquad P(A_i) = \frac{1}{n}$$

$$= 1 - \prod_{i=1}^{n} \left(1 - \frac{1}{n}\right) = 1 - \left(1 - \frac{1}{n}\right)^n = 1 - e^{-1}$$

$$\boxed{\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e}$$

$$\lim_{n \to \infty} \left(1 - \frac{1}{n}\right)^n = e^{-1}$$

$$\prod_{i=1}^{n} f(i) = f(1) \times f(2) \times \cdots \times f(n)$$

$$\sum_{i=1}^{n} f(i) = f(1) + f(2) + \cdots + f(n)$$