Section 3.10 Poisson Distribution

Tuesday, October 4, 2016 4:07 PM

Section 3.10 Poiss...

- **Binomial**: Here, $n = 20$ and $p = r/N = 0.10$. Thus,

$$P(X \geq 4) = 1 - P(X \leq 3) = 1 - \sum_{x=0}^{3} \binom{20}{x} (0.1)^x (0.9)^{20-x} \approx 0.132953. \ \square$$

REMARK: Of course, since the binomial and hypergeometric models are similar when N is large, their means and variances are similar too. Note the similarities; recall that the quantity $r/N \to p$, as $N \to \infty$:

$$E(Y) = n\left(\frac{r}{N}\right) \approx np$$

and

$$V(Y) = n\left(\frac{r}{N}\right)\left(\frac{N-r}{N}\right)\left(\frac{N-n}{N-1}\right) \approx np(1-p).$$

3.10 Poisson distribution

TERMINOLOGY: Let the number of occurrences in a given continuous interval of time or space be counted. A **Poisson process** enjoys the following properties:

(1) the number of occurrences in non-overlapping intervals are **independent** random variables.

(2) The probability of an occurrence in a sufficiently short interval is **proportional to the length** of the interval.

(3) The probability of 2 or more occurrences in a sufficiently short interval is zero.

GOAL: Suppose that a process satisfies the above three conditions, and let Y denote the number of occurrences in an interval of length one. Our goal is to find an expression for $p_Y(y) = P(Y = y)$, the pmf of Y.

APPROACH: Envision partitioning the unit interval $[0, 1]$ into n subintervals, each of size $1/n$. Now, if n is sufficiently large (i.e., much larger than y), then we can approximate the probability that y events occur in this unit interval by finding the probability that exactly one event (occurrence) occurs in exactly y of the subintervals.

- By Property (2), we know that the probability of one event in any one subinterval is **proportional** to the subinterval's length, say λ/n, where λ is the proportionality constant.

- By Property (3), the probability of more than one occurrence in any subinterval is zero (for n large).

- Consider the occurrence/non-occurrence of an event in each subinterval as a **Bernoulli trial**. Then, by Property (1), we have a sequence of n Bernoulli trials, each with probability of "success" $p = \lambda/n$. Thus, a binomial (approximate) calculation gives

$$P(Y=y) \approx \binom{n}{y}\left(\frac{\lambda}{n}\right)^y\left(1-\frac{\lambda}{n}\right)^{n-y}.$$

To improve the approximation for $P(Y=y)$, we let n get large without bound. Then,

$$\begin{aligned}
\lim_{n\to\infty} P(Y=y) &= \lim_{n\to\infty}\binom{n}{y}\left(\frac{\lambda}{n}\right)^y\left(1-\frac{\lambda}{n}\right)^{n-y}\\
&= \lim_{n\to\infty}\frac{n!}{y!(n-y)!}\lambda^y\left(\frac{1}{n}\right)^y\left(1-\frac{\lambda}{n}\right)^n\left(\frac{1}{1-\frac{\lambda}{n}}\right)^y\\
&= \lim_{n\to\infty}\underbrace{\frac{n(n-1)\cdots(n-y+1)}{n^y}}_{a_n}\underbrace{\frac{\lambda^y}{y!}}_{b_n}\underbrace{\left(1-\frac{\lambda}{n}\right)^n}_{c_n}\underbrace{\left(\frac{1}{1-\frac{\lambda}{n}}\right)^y}_{d_n}.
\end{aligned}$$

Now, the limit of the product is the product of the limits:

$$\begin{aligned}
\lim_{n\to\infty} a_n &= \lim_{n\to\infty}\frac{n(n-1)\cdots(n-y+1)}{n^y} = 1\\
\lim_{n\to\infty} b_n &= \lim_{n\to\infty}\frac{\lambda^y}{y!} = \frac{\lambda^y}{y!}\\
\lim_{n\to\infty} c_n &= \lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n = e^{-\lambda}\\
\lim_{n\to\infty} d_n &= \lim_{n\to\infty}\left(\frac{1}{1-\frac{\lambda}{n}}\right)^y = 1.
\end{aligned}$$

We have shown that

$$\lim_{n\to\infty} P(Y=y) = \frac{\lambda^y e^{-\lambda}}{y!}.$$

POISSON PMF: A discrete random variable Y is said to follow a **Poisson distribution** with rate λ if the pmf of Y is given by

$$p_Y(y) = \begin{cases} \frac{\lambda^y e^{-\lambda}}{y!}, & y = 0, 1, 2, \ldots \\ 0, & \text{otherwise.} \end{cases}$$

$$\sum_{y=0}^{\infty} \frac{\lambda^y}{y!} = e^{\lambda}$$

We write $Y \sim \text{Poisson}(\lambda)$.

NOTE: Clearly $p_Y(y) > 0$ for all $y \in R$. That $p_Y(y)$ sums to one is easily seen as

$$\begin{aligned} \sum_{y \in R} p_Y(y) &= \sum_{y=0}^{\infty} \frac{\lambda^y e^{-\lambda}}{y!} \\ &= e^{-\lambda} \sum_{y=0}^{\infty} \frac{\lambda^y}{y!} = e^{-\lambda} e^{\lambda} = 1, \end{aligned}$$

since $\sum_{y=0}^{\infty} \lambda^y / y!$ is the McLaurin series expansion of e^{λ}. □

Poisson !!!

$\boxed{\text{\# of occurrences in a unit}}$

EXAMPLES: Discrete random variables that might be modeled using a Poisson distribution include

(1) the number of customers entering a post office in a given day.

(2) the number of α-particles discharged from a radioactive substance in one second.

(3) the number of machine breakdowns per month.

(4) the number of blemishes on a piece of artificial turf.

(5) the number of chocolate chips in a Chips-Ahoy cookie.

Example 3.22. The number of cars Y abandoned weekly on a highway is modeled using a Poisson distribution with $\lambda = 2.2$. In a given week, what is the probability that

(a) no cars are abandoned?

(b) exactly one car is abandoned?

(c) at most one car is abandoned?

(d) at least one car is abandoned?

SOLUTIONS. We have $Y \sim \text{Poisson}(\lambda = 2.2)$.

(a)

$$P(Y=0) = p_Y(0) = \frac{(2.2)^0 e^{-2.2}}{0!} = e^{-2.2} = 0.1108$$ → poissonpdf(2.2, 0)

(b)

$$P(Y=1) = p_Y(1) = \frac{(2.2)^1 e^{-2.2}}{1!} = 2.2e^{-2.2} = 0.2438$$ → poissonpdf(2.2, 1)

(c) $P(Y \leq 1) = P(Y=0) + P(Y=1) = p_Y(0) + p_Y(1) = 0.1108 + 0.2438 = 0.3456$ → poissoncdf(2.2, 1)

(d) $P(Y \geq 1) = 1 - P(Y=0) = 1 - p_Y(0) = 1 - 0.1108 = 0.8892$. □ → 1 − poissonpdf(2.2, 0)

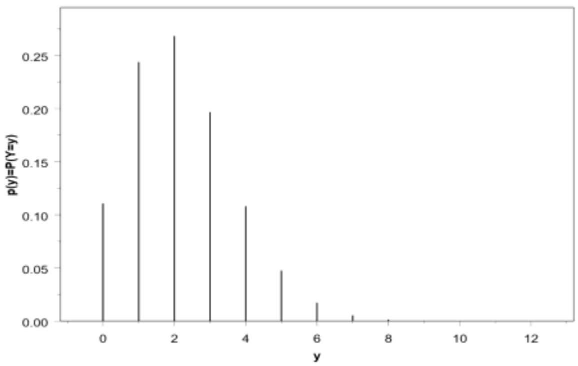

Figure 3.4: *Probability histogram for the number of abandoned cars. This represents the* Poisson($\lambda = 2.2$) *model in Example* 3.22.

REMARK: WMS's Appendix III, (Table 3, pp 843-847) includes an impressive table for Poisson probabilities of the form

$$F_Y(a) = P(Y \leq a) = \sum_{y=0}^{a} \frac{\lambda^y e^{-\lambda}}{y!}.$$

Recall that this function is called the **cumulative distribution function** of Y. This makes computing compound event probabilities much easier.

POISSON MGF: Suppose that $Y \sim \text{Poisson}(\lambda)$. The mgf of Y, for all t, is given by

$$\begin{aligned} m_Y(t) = E(e^{tY}) &= \sum_{y=0}^{\infty} e^{ty}\left(\frac{\lambda^y e^{-\lambda}}{y!}\right) \\ &= e^{-\lambda}\underbrace{\sum_{y=0}^{\infty}\frac{(\lambda e^t)^y}{y!}}_{=\exp(\lambda e^t)} \\ &= e^{-\lambda}e^{\lambda e^t} = \exp[\lambda(e^t-1)]. \ \square \end{aligned}$$

since $e^x = \sum_{y=0}^{\infty} \frac{x^y}{y!}$ for any x.

MEAN AND VARIANCE: With the mgf, we can derive the mean and variance. Differentiating the mgf, we get

$$m_Y'(t) = \frac{d}{dt}m_Y(t) = \frac{d}{dt}\exp[\lambda(e^t-1)] = \lambda e^t \exp[\lambda(e^t-1)].$$

Thus,

$$E(Y) = \frac{d}{dt}m_Y(t)\bigg|_{t=0} = \lambda e^0 \exp[\lambda(e^0-1)] = \lambda.$$

Now, we need to find the second moment. Using the product rule, we have

$$\begin{aligned} \frac{d^2}{dt^2}m_Y(t) &= \frac{d}{dt}\underbrace{\lambda e^t \exp[\lambda(e^t-1)]}_{m_Y'(t)} \\ &= \lambda e^t \exp[\lambda(e^t-1)] + (\lambda e^t)^2 \exp[\lambda(e^t-1)]. \end{aligned}$$

Thus,

$$E(Y^2) = \frac{d^2}{dt^2}m_Y(t)\bigg|_{t=0} = \lambda e^0 \exp[\lambda(e^0-1)] + (\lambda e^0)^2\exp[\lambda(e^0-1)] = \lambda + \lambda^2$$

so that

$$\begin{aligned} V(Y) &= E(Y^2) - [E(Y)]^2 \\ &= \lambda + \lambda^2 - \lambda^2 = \lambda. \ \square \end{aligned}$$

REVELATION: The mean and variance of a Poisson random variable are always equal. $= \lambda$

Example 3.23. Suppose that Y denotes the number of defects observed in one month at an automotive plant. From past experience, engineers believe that a Poisson model is appropriate and that $E(Y) = \lambda = 7$ defects/month.

QUESTION 1: What is the probability that, in a given month, we observe 11 or more defects?

SOLUTION. We want to compute

$$P(Y \geq 11) = 1 - \underbrace{P(Y \leq 10)}_{\text{Table 3}} = 1 - 0.901 = 0.099.$$

QUESTION 2: What is the probability that, in a given year, we have two or more months with 11 or more defects?

SOLUTION. First, we assume that the 12 months are independent (is this reasonable?), and call the event $A = \{11 \text{ or more defects in a month}\}$ a "success." Thus, under our independence assumptions and viewing each month as a "trial," we have a sequence of 12 Bernoulli trials with "success" probability $p = P(A) = 0.099$. Let X denote the number of months where we observe 11 or more defects. Then, $X \sim b(12, 0.099)$ and

$$\begin{aligned} P(X \geq 2) &= 1 - P(X = 0) - P(X = 1) \\ &= 1 - \binom{12}{0}(0.099)^0(1-0.099)^{12} - \binom{12}{1}(0.099)^1(1-0.099)^{11} \\ &= 1 - 0.2862 - 0.3774 = 0.3364. \ \square \end{aligned}$$

Important, involves both binomial and poisson

POISSON PROCESSES OF ARBITRARY LENGTH: If events or occurrences in a Poisson process occur at a rate of λ per unit time or space, then the number of occurrences in an interval of length t follows a Poisson distribution with mean λt.

consistency in units
very important property for Poisson!!!

Example 3.24. Phone calls arrive at a call center according to a Poisson process, at a rate of $\lambda = 3$ per minute. If Y represents the number of calls received in 5 minutes, then $Y \sim \text{Poisson}(15)$. The probability that 8 or fewer calls come in during a 5-minute span is

$$P(Y \leq 8) = \sum_{y=0}^{8} \frac{15^y e^{-15}}{y!} = 0.037,$$

using Table 3 (WMS). $\square$

$\hookrightarrow$ poissoncdf(15, 8)

POISSON-BINOMIAL LINK: We have seen that the hypergeometric and binomial models are related; as it turns out, so are the Poisson and binomial models. This should not be surprising because we derived the Poisson pmf by appealing to a binomial approximation.

RELATIONSHIP: Suppose that $Y \sim b(n,p)$. If n is large and p is small, then

$$p_Y(y) = \binom{n}{y} p^y (1-p)^{n-y} \approx \frac{\lambda^y e^{-\lambda}}{y!},$$

for $y \in R = \{0, 1, 2, ..., n\}$, where $\lambda = np$.

Example 3.25. Hepatitis C (HCV) is a viral infection that causes cirrhosis and cancer of the liver. Since HCV is transmitted through contact with infectious blood, screening donors is important to prevent further transmission. The World Health Organization has projected that HCV will be a major burden on the US health care system before the year 2020. For public-health reasons, researchers take a sample of $n = 1875$ blood donors and screen each individual for HCV. If 3 percent of the entire population is infected, what is the probability that 50 or more are HCV-positive?

SOLUTION. Let Y denote the number of HCV-infected individuals in our sample. We compute the probability $P(Y \geq 50)$ using both the binomial and Poisson models.

- **Binomial**: Here, $n = 1875$ and $p = 0.03$. Thus,

$$P(Y \geq 50) = \sum_{y=50}^{1875} \binom{1875}{y} (0.03)^y (0.97)^{1875-y} \approx 0.818783.$$

- **Poisson**: Here, $\lambda = np = 1875(0.03) \approx 56.25$. Thus,

$$P(Y \geq 50) = \sum_{y=50}^{\infty} \frac{(56.25)^y e^{-56.25}}{y!} \approx 0.814932.$$

As we can see, the Poisson approximation is quite good. $\square$

RELATIONSHIP: One can see that the hypergeometric, binomial, and Poisson models are related in the following way:

$$\text{hyper}(N, n, r) \longleftrightarrow b(n,p) \longleftrightarrow \text{Poisson}(\lambda).$$

The first link results when N is large and $r/N \to p$. The second link results when n is large and p is small so that $\lambda/n \to p$. When these situations are combined, as you might suspect, one can approximate the hypergeometric model with a Poisson model!