Section 3.2 Probability distributions for discrete random varaibles

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NOTE: From these probability calculations; note that we can

- work on the sample space S and compute probabilities from S, or
- work on $\mathcal{R}$ and compute probabilities for events $\{Y \in B\}$, where $B \subset \mathcal{R}$.

NOTATION: We denote a random variable Y using a capital letter. We denote an observed value of Y by y, a lowercase letter. **This is standard notation.** For example, if Y denotes the weight (in ounces) of the next newborn boy in Columbia, SC, then Y is random variable. After the baby is born, we observe that the baby weighs $y = 128$ oz.

3.2 Probability distributions for discrete random variables

TERMINOLOGY: The **support** of a random variable Y is set of all possible values that Y can assume. We will denote the support set by R.

TERMINOLOGY: If the random variable Y has a support set R that is countable (finitely or infinitely), we call Y a **discrete** random variable.

Example 3.2. An experiment consists of rolling an unbiased die. Consider the two random variables:

$$\begin{aligned} X &= \text{face value on the first roll} \\ Y &= \text{number of rolls needed to observe a six.} \end{aligned}$$

finite

The support of X is $R_X = \{1, 2, 3, 4, 5, 6\}$. The support of Y is $R_Y = \{1, 2, 3, ...\}$. R_X is finitely countable and R_Y is infinitely countable; thus, both X and Y are discrete. □

GOAL: For a discrete random variable Y, we would like to find $P(Y = y)$ for any $y \in R$. Mathematically,

$$p_Y(y) \equiv P(Y = y) = \sum P[E_i \in S : Y(E_i) = y],$$

R

for all $y \in R$.

Previous Ex.

Y: # of heads

$P(Y=1)$

$= P((H,T)) + P((T,H))$

$= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

TERMINOLOGY: Suppose that Y is a discrete random variable. The function $p_Y(y) = P(Y = y)$ is called the **probability mass function** (**pmf**) for Y. The pmf $p_Y(y)$ consists of two parts:

(a) R, the support set of Y

(b) a probability assignment $P(Y = y)$, for all $y \in R$.

$P_Y(y)$

PROPERTIES: A pmf $p_Y(y)$ for a discrete random variable Y satisfies the following:

(1) $p_Y(y) > 0$, for all $y \in R$ [NOTE: if $y \notin R$, then $p_Y(y) = 0$]

(2) The sum of the probabilities, taken over all support points, must equal one; i.e.,

$$\sum_{y \in R} p_Y(y) = 1.$$

IMPORTANT: Suppose that Y is a **discrete** random variable. The probability of an event $\{Y \in B\}$ is computed by adding the probabilities $p_Y(y)$ for all $y \in B$; i.e.,

$$P(Y \in B) = \sum_{y \in B} p_Y(y).$$

Example 3.3. An experiment consists of rolling two fair dice and observing the face on each. The sample space consists of $6 \times 6 = 36$ sample points:

$$\begin{aligned} S = \{&\overset{2}{(1,1)}, (1,2), (1,3), (1,4), (1,5), (1,6), (2,1), (2,2), (2,3), (2,4), (2,5), (2,6),\\ &(3,1), (3,2), (3,3), (3,4), (3,5), (3,6), (4,1), (4,2), (4,3), (4,4), (4,5), (4,6),\\ &(5,1), (5,2), (5,3), (5,4), (5,5), (5,6), (6,1), (6,2), (6,3), (6,4), (6,5), \underset{12}{(6,6)}\}. \end{aligned}$$

Let the random variable Y record the sum of the two faces. Note that $R = \{2, 3, ..., 12\}$. We now compute the probability associated with each support point $y \in R$:

$$\begin{aligned} P(Y = 2) &= P(\{\text{all } E_i \in S \text{ where } Y(E_i) = y = 2\})\\ &= P[(1,1)] = 1/36. \end{aligned}$$

$$\begin{aligned} P(Y=3) &= P(\{\text{all } E_i \in S \text{ where } Y(E_i)=y=3\}) \\ &= P[(1,2)] + P[(2,1)] = 2/36. \end{aligned}$$

The calculation $P(Y=y)$ is performed similarly for $y=4,5,...,12$. The pmf for Y can be given as a formula, a table, or a graph. In tabular form, the pmf of Y is given by

y	2	3	4	5	6	7	8	9	10	11	12
$p_Y(y)$	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

pmf of Y

A **probability histogram** is a display which depicts a pmf in graphical form. In this example, the probability histogram looks like

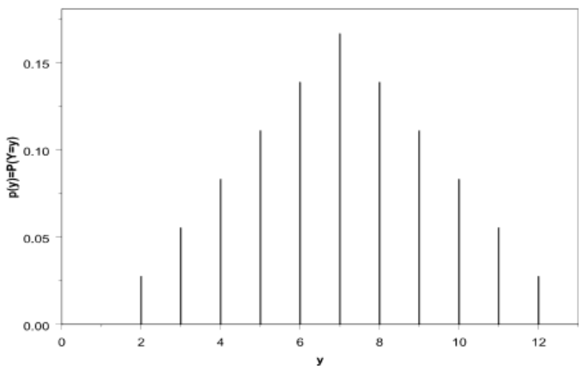


figure.

$$\sum_{y=2}^{12} p_Y(y) = 1$$

A closed-form formula for the pmf exists and is given by

$$p_Y(y) = \begin{cases} \frac{1}{36}(6-|7-y|), & y=2,3,...,12 \\ 0, & \text{otherwise.} \end{cases}$$

Define the event $B=\{3,5,7,9,11\}$; i.e., the sum Y is odd. We have

$$\begin{aligned} P(Y\in B) = \sum_{y\in B} p_Y(y) &= p_Y(3)+p_Y(5)+p_Y(7)+p_Y(9)+p_Y(11) \\ &= 2/36+4/36+6/36+4/36+2/36 = 1/2. \ \square \end{aligned}$$

Example 3.4. An experiment consists of rolling an unbiased die until the first "6" is observed. Let Y denote the number of rolls needed. The support is $R = \{1, 2, ...\}$. Assuming independent trials, we have

$$\begin{aligned} P(Y=1) &= \frac{1}{6} \\ P(Y=2) &= \frac{5}{6}\times\frac{1}{6} \\ P(Y=3) &= \frac{5}{6}\times\frac{5}{6}\times\frac{1}{6}; \end{aligned}$$

$P(\text{"six"}) = \frac{1}{6}$

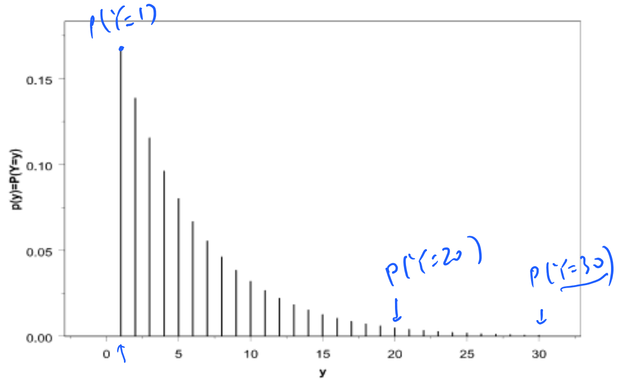
$P(\text{not six}) = 1 - \frac{1}{6} = \frac{5}{6}$

Recognizing the pattern, we see that the pmf for Y is given by

$$p_Y(y) = \begin{cases} \frac{1}{6}\left(\frac{5}{6}\right)^{y-1}, & y = 1, 2, ... \\ 0, & \text{otherwise.} \end{cases}$$

$P(Y=y) = \left(\frac{5}{6}\right)^{y-1}\frac{1}{6} \geq 0$

This pmf is depicted in a probability histogram below:



QUESTION: Is this a **valid** pmf; i.e., do the probabilities $p_Y(y)$ sum to one? Note that

$x = y-1$

$y=1$, $y-1=0$

$y=2$, $y-1=1$

$$\begin{aligned} ?\ 1 = \sum_{y\in R} p_Y(y) &= \sum_{y=1}^{\infty}\frac{1}{6}\left(\frac{5}{6}\right)^{y-1} \\ &= \sum_{x=0}^{\infty}\frac{1}{6}\left(\frac{5}{6}\right)^{x} \\ &= \left(\frac{\frac{1}{6}}{1-\frac{5}{6}}\right) = 1. \square \end{aligned}$$

$\sum_{x=0}^{\infty} ar^x = \frac{a}{1-r}$ if $|r|<1$?

IMPORTANT: In the last calculation, we have used an important fact concerning **infinite geometric series**; namely, if a is any real number and $|r| < 1$. Then,

$$\sum_{x=0}^{\infty} ar^x = \frac{a}{1-r}.$$

We will use this fact many times in this course!

EXERCISE: Find the probability that the first "6" is observed on (a) an odd-numbered roll (b) an even-numbered roll. Which event is more likely? ☐

$6/11$ $5/11$ (a) is more likely

3.3 Mathematical expectation

TERMINOLOGY: Let Y be a discrete random variable with pmf $p_Y(y)$ and support R. The **expected value** of Y is given by

$$E(Y) = \sum_{y \in R} y p_Y(y).$$

The expected value for discrete random variable Y is simply a weighted average of the possible values of Y. Each support point y is weighted by the probability $p_Y(y)$.

ASIDE: When R is a countably infinite set, then the sum $\sum_{y \in R} y p_Y(y)$ may not exist (not surprising since sometimes infinite series do diverge). Mathematically, we require the sum above to be **absolutely convergent**; i.e.,

$$\sum_{y \in R} |y| p_Y(y) < \infty.$$

If this is true, we say that $E(Y)$ exists. If this is not true, then we say that $E(Y)$ does not exist. NOTE: If R is a finite set, then $E(Y)$ always exists, because a finite sum of finite quantities is always finite.

Example 3.5. Let the random variable Y have pmf

$$p_Y(y) = \begin{cases} \frac{1}{10}(5-y), & y = 1, 2, 3, 4 \\ 0, & \text{otherwise.} \end{cases}$$

$P((a)) = P($the first "6" is on an odd-numbered roll$)$

$= P(Y = 1, 3, 5, 7, 9, \ldots)$

$= \sum_{Y=1,3,5,7,\ldots} P_Y(y)$

$= \sum_{k=0}^{\infty} P_Y(2k+1)$

$= \sum_{k=0}^{\infty} \left(\frac{5}{6}\right)^{2k+1-1} \frac{1}{6}$

$= \sum_{k=0}^{\infty} \left(\frac{5}{6}\right)^{2k} \times \frac{1}{6}$

$= \sum_{k=0}^{\infty} \left(\frac{25}{36}\right)^{k} \times \frac{1}{6}$

$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$

$= \frac{1}{6} \Big/ \left(1 - \frac{25}{36}\right) = \frac{6}{11}$