Section 3.3 Mathematical expectation

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Section 3.3 Mathema...

IMPORTANT: In the last calculation, we have used an important fact concerning **infinite geometric series**; namely, if a is any real number and $|r| < 1$. Then,

$$\sum_{x=0}^{\infty} ar^x = \frac{a}{1-r}.$$

We will use this fact many times in this course!

EXERCISE: Find the probability that the first "6" is observed on (a) an odd-numbered roll (b) an even-numbered roll. Which event is more likely? □

3.3 Mathematical expectation

TERMINOLOGY: Let Y be a discrete random variable with pmf $p_Y(y)$ and support R. The **expected value** of Y is given by

$$E(Y) = \underline{\sum_{y \in R} y p_Y(y)}.$$

The expected value for discrete random variable Y is simply a weighted average of the possible values of Y. Each support point y is weighted by the probability $p_Y(y)$.

ASIDE: When R is a countably infinite set, then the sum $\sum_{y \in R} y p_Y(y)$ may not exist (not surprising since sometimes infinite series do diverge). Mathematically, we require the sum above to be **absolutely convergent**; i.e.,

$$\sum_{y \in R} |y| p_Y(y) < \infty.$$

guarantees $\sum_{y \in R} y P_Y(y) < +\infty$

If this is true, we say that $E(Y)$ exists. If this is not true, then we say that $E(Y)$ does not exist. NOTE: If R is a finite set, then $E(Y)$ always exists, because a finite sum of finite quantities is always finite.

Example 3.5. Let the random variable Y have pmf

$$\Big\{ \ 1(5-y), \quad y = 1, 2, 3, 4$$

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$$p_Y(y) = \begin{cases} \frac{1}{10}(5-y), & y = 1,2,3,4 \\ 0, & \text{otherwise.} \end{cases}$$

The expected value of Y is given by

$$E(Y) = \sum_{y \in R} y p_Y(y) = \sum_{y=1}^{4} y\left[\frac{1}{10}(5-y)\right] = 1(4/10) + 2(3/10) + 3(2/10) + 4(1/10) = 2. \square$$

$1 \times \frac{4}{10}$

INTERPRETATION: The quantity $E(Y)$ has many interpretations:

(a) the "center of gravity" of a probability distribution

(b) a long-run average

(c) the **first moment** of the random variable $E[Y]$ $E[Y^2]$ $E[Y^3]$

(d) the mean of a population.

FUNCTIONS OF Y: Let Y be a discrete random variable with pmf $p_Y(y)$ and support R. Suppose that g is a real-valued function. Then, $g(Y)$ is a random variable and

$$E[g(Y)] = \sum_{y \in R} g(y) p_Y(y).$$

The proof of this result is given on pp 93 (WMS). Again, we require that

$$\sum_{y \in R} |g(y)| p_Y(y) < \infty.$$

If this is not true, then $E[g(Y)]$ does not exist.

Example 3.6. In Example 3.5, find $E(Y^2)$ and $E(e^Y)$.

SOLUTION: The functions $g_1(Y) = Y^2$ and $g_2(Y) = e^Y$ are real functions of Y. From the definition, we have

$$\begin{aligned} E(Y^2) &= \sum_{y \in R} y^2 p_Y(y) \\ &= \sum_{y=1}^{4} y^2 \left[\frac{1}{10}(5-y)\right] = 1^2(4/10) + 2^2(3/10) + 3^2(2/10) + 4^2(1/10) = 5. \end{aligned}$$

$$= \sum_{y=1}^{4} y^2 \left[\frac{1}{10}(5-y)\right] = 1^2(4/10) + 2^2(3/10) + 3^2(2/10) + 4^2(1/10) = 5.$$

Also,

$$\begin{aligned} E(e^Y) &= \sum_{y \in R} e^y p_Y(y) \\ &= \sum_{y=1}^{4} e^y \left[\frac{1}{10}(5-y)\right] = e^1(4/10) + e^2(3/10) + e^3(2/10) + e^4(1/10) \approx 12.78. \ \square \end{aligned}$$

Example 3.7. *The discrete uniform distribution.* Suppose that the random variable X has pmf

$$p_X(x) = \begin{cases} 1/m, & x = 1, 2, ..., m \\ 0, & \text{otherwise,} \end{cases}$$

where m is a positive integer larger than 1. Find the expected value of X.

SOLUTION. The expected value of X is given by

$$E(X) = \sum_{x \in R} x p_X(x) = \sum_{x=1}^{m} x\left(\frac{1}{m}\right) = \frac{1}{m}\sum_{x=1}^{m} x = \frac{1}{m}\left[\frac{m(m+1)}{2}\right] = \frac{m+1}{2}.$$

$1+2+3+4+5+6+ \cdots + m = \frac{m(m+1)}{2}$

We have used the well-known fact that $\sum_{x=1}^{m} x = m(m+1)/2$; this can be proven by induction. If $m = 6$, then the discrete uniform distribution serves as a probability model for the outcome of an unbiased die:

x	1	2	3	4	5	6
$p_X(x)$	1/6	1/6	1/6	1/6	1/6	1/6

The expected value of X is $E(X) = (6+1)/2 = 3.5$. $\square$

PROPERTIES OF EXPECTATIONS: Let Y be a discrete random variable with pmf $p_Y(y)$ and support R. Suppose that $g, g_1, g_2, ..., g_k$ are real-valued functions, and let c be any real constant. Expectations satisfy the following (linearity) properties:

(a) $E(c) = c$

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(b) $E[cg(Y)] = cE[g(Y)]$

(c) $E[\sum_{j=1}^{k} g_j(Y)] = \sum_{j=1}^{k} E[g_j(Y)]$.

Example 3.8. In a one-hour period, the number of gallons of a certain toxic chemical that is produced at a local plant, say Y, has the following pmf:

y	0	1	2	3
$p_Y(y)$	0.2	0.3	0.3	0.2

(a) Compute the expected number of gallons produced during a one-hour period.

(b) The cost (in hundreds of dollars) to produce Y gallons is given by the cost function $C(Y) = 3 + 12Y + 2Y^2$. What is the expected cost in a one-hour period?

SOLUTION: (a) The expected value of Y is

$$E(Y) = \sum_{y \in R} y p_Y(y) = 0(0.2) + 1(0.3) + 2(0.3) + 3(0.2) = 1.5.$$

That is, we would expect 1.5 gallons of the toxic chemical to be produced per hour. For (b), we first compute $E(Y^2)$:

$$E(Y^2) = \sum_{y \in R} y^2 p_Y(y) = 0^2(0.2) + 1^2(0.3) + 2^2(0.3) + 3^2(0.2) = 3.3.$$

Finally,

$$\begin{aligned} E[C(Y)] &= E(3 + 12Y + 2Y^2) \\ &= 3 + 12E(Y) + 2E(Y^2) = 3 + 12(1.5) + 2(3.3) = 27.6. \end{aligned}$$

The expected hourly cost is \$2,760.00. $\square$

The expected hourly cost is \$2,700.00. $\square$

3.4 Variance

TERMINOLOGY: Let Y be a discrete random variable with pmf $p_Y(y)$, support R, and expected value $E(Y) = \mu$. The **variance** of Y is given by

$$\sigma^2 \equiv V(Y) \equiv E[(Y-\mu)^2] = \sum_{y \in R} (y-\mu)^2 p_Y(y).$$

The **standard deviation** of Y is given by the positive square root of the variance; i.e.,

$$\sigma = \sqrt{\sigma^2} = \sqrt{V(Y)}.$$

FACTS: The variance σ^2 satisfies the following:

(a) $\sigma^2 \geq 0$.