Section 4.4 Mathematical Expectations

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Section 4.4
Mathema...

Using integration by parts with $u = cy$ and $dv = e^{-y/2}dy$, we have

$$1 = \int_0^\infty cye^{-y/2}dy = \underbrace{-2cye^{-y/2}\Big|_0^\infty}_{=0} + \int_0^\infty 2ce^{-y/2}dy$$

$$= 2c(-2)e^{-y/2}\Big|_0^\infty = 0 - (-4c) = 4c.$$

Solving for c, we get $c = 1/4$. $\square$

QUANTILES: Suppose that Y is a continuous random variable with cdf $F_Y(y)$ and let $0 < p < 1$. The p**th quantile** of the distribution of Y, denoted by ϕ_p, solves

$$F_Y(\phi_p) = P(Y \leq \phi_p) = \int_{-\infty}^{\phi_p} f_Y(y)dy = p.$$

The **median** of the distribution of Y is the $p = 0.5$ quantile. That is, the median $\phi_{0.5}$ solves

$$F_Y(\phi_{0.5}) = P(Y \leq \phi_{0.5}) = \int_{-\infty}^{\phi_{0.5}} f_Y(y)dy = 0.5.$$

Another name for the pth quantile is the $100p$**th percentile**.

EXERCISE. Find the median of Y in Examples 4.3, 4.4, and 4.5.

REMARK: For Y discrete, there are some potential problems with the definition that ϕ_p solves $F_Y(\phi_p) = P(Y \leq \phi_p) = p$. The reason is that there may be many values of ϕ_p that satisfy this equation. For example, in Example 4.1, it is easy to see that the median $\phi_{0.5} = 0$ because $F_Y(0) = P(Y \leq 0) = 0.5$. However, $\phi_{0.5} = 0.5$ also satisfies $F_Y(\phi_{0.5}) = 0.5$. By convention, in discrete distributions, the pth quantile ϕ_p is taken to be the smallest value satisfying $F_Y(\phi_p) = P(Y \leq \phi_p) \geq p$.

4.4 Mathematical expectation

If Y is a discrete r.v. with pmf $f_Y(y)$, support R

4.4.1 Expected value

? $E(Y) = \sum_R y f_Y(y)$

TERMINOLOGY: Let Y be a continuous random variable with pdf $f_Y(y)$ and support R. The **expected value** of Y is given by

$$E(Y) = \int_R y f_Y(y)dy.$$

Mathematically, we require that

$$\int_R |y| f_Y(y)dy < \infty.$$

If this is not true, we say that $E(Y)$ does not exist. If g is a real-valued function, then $g(Y)$ is a random variable and

$$E[g(Y)] = \int_R g(y) f_Y(y)dy,$$

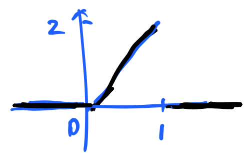
provided that this integral exists.

Discrete $E[g(Y)] = \sum_{y \in R} g(y) f(y)$

Example 4.6. Suppose that Y has pdf given by

$R = (0,1)$

$$f_Y(y) = \begin{cases} 2y, & 0 < y < 1 \\ 0, & \text{otherwise.} \end{cases}$$



Find $E(Y)$, $E(Y^2)$, and $E(\ln Y)$.

SOLUTION. The expected value of Y is given by

$E(Y) = \int_R y f_Y(y) dy$

$R = \{y : f_Y(y) > 0\}$

$$\begin{aligned} E(Y) &= \int_0^1 y f_Y(y)dy \\ &= \int_0^1 y(2y)dy \\ &= \int_0^1 2y^2 dy = 2\left(\frac{y^3}{3}\Big|_0^1\right) = 2\left(\frac{1}{3} - 0\right) = 2/3. \end{aligned}$$

?

The second moment is

$g(Y) = Y^2$

$E[g(Y)]$

$$\begin{aligned} E(Y^2) &= \int_0^1 y^2 f_Y(y)dy \\ &= \int_0^1 y^2(2y)dy \\ &= \int_0^1 2y^3 dy = 2\left(\frac{y^4}{4}\Big|_0^1\right) = 2\left(\frac{1}{4} - 0\right) = 1/2. \end{aligned}$$

?

Finally,

$$E(\ln Y) = \int_0^1 \ln y(2y)dy.$$

$g(Y) = \ln Y$

To solve this integral, use integration by parts with $u = \ln y$ and $dv = 2ydy$:

$$E(\ln Y) = \underbrace{y^2 \ln y\Big|_0^1}_{=0} - \int_0^1 ydy = -\left(\frac{y^2}{2}\Big|_0^1\right) = -\frac{1}{2}. \quad \square$$

??

$\int_0^1 \ln y \, 2y \, dy$

$= \int_0^1 u \, dv$

$= uv\Big|_0^1 - \int_0^1 v \, du$

$= \ln y \times y^2 \Big|_0^1 - \int_0^1 y^2 \, d\ln y$

$= (\ln y \times y^2)\Big|_0^1 - \int_0^1 y^2 \times \frac{1}{y} dy$

$u = \ln y$

$dv = 2y dy$

$v = y^2$

$d \ln y = \frac{1}{y} dy$

PROPERTIES OF EXPECTATIONS: Let Y be a continuous random variable with pdf $f_Y(y)$ and support R, suppose that $g, g_1, g_2, ..., g_k$ are real-valued functions, and let c be any real constant. Then,

$$E[c\,g(Y)] = \int_R c\,g(y) f_Y(y)\,dy$$
$$= c \int_R g(y) f_Y(y)\,dy$$
$$= c\,E[g(Y)]$$

(a) $E(c) = c$

(b) $E[cg(Y)] = cE[g(Y)]$

(c) $E[\sum_{j=1}^{k} g_j(Y)] = \sum_{j=1}^{k} E[g_j(Y)]$.

These properties are identical to those we discussed in the discrete case.

4.4.2 Variance

$$V(Y) = E(Y^2) - [E(Y)]^2$$

TERMINOLOGY: Let Y be a continuous random variable with pdf $f_Y(y)$, support R, and mean $E(Y) = \mu$. The **variance** of Y is given by

$$\sigma^2 \equiv V(Y) \equiv E[(Y-\mu)^2] = \int_R (y-\mu)^2 f_Y(y)dy. \geq 0$$

HW problem

The variance computing formula still applies in the continuous case, that is,

$$V(Y) = E(Y^2) - [E(Y)]^2. = \int_R y^2 f_Y(y)\,dy - \left(\int_R y f_Y(y)\,dy\right)^2$$

Example 4.7. Suppose that Y has pdf given by

$$f_Y(y) = \begin{cases} 2y, & 0 < y < 1 \\ 0, & \text{otherwise.} \end{cases}$$

Find $\sigma^2 = V(Y)$.

SOLUTION. We computed $E(Y) = \mu = 2/3$ in Example 4.6. Using the definition above,

$$V(Y) = \int_0^1 \left(y - \frac{2}{3}\right)^2 (2y)dy.$$

Instead of doing this integral, it is easier to use the variance computing formula $V(Y) = E(Y^2) - [E(Y)]^2$. In Example 4.6, we computed the second moment $E(Y^2) = 1/2$. Thus,

$$V(Y) = E(Y^2) - [E(Y)]^2 = \frac{1}{2} - \left(\frac{2}{3}\right)^2 = 1/18. \quad \square$$

4.4.3 Moment generating functions

TERMINOLOGY: Let Y be a continuous random variable with pdf $f_Y(y)$ and support R. The **moment generating function (mgf)** for Y, denoted by $m_Y(t)$, is given by

$$m_Y(t) = E(e^{tY}) = \int_R e^{ty} f_Y(y)dy,$$

provided $E(e^{tY}) < \infty$ for all t in an open neighborhood about 0; i.e., there exists some $h > 0$ such that $E(e^{tY}) < \infty$ for all $t \in (-h, h)$. If $E(e^{tY})$ does not exist in an open neighborhood of 0, we say that the moment generating function does not exist.

Example 4.8. Suppose that the pdf of Y is given by

$$f_Y(y) = \begin{cases} e^{-y}, & y > 0 \\ 0, & \text{otherwise.} \end{cases}$$

$R = (0, +\infty)$

Find the mgf of Y and use it to compute $E(Y)$ and $V(Y)$.

SOLUTION.

$$\begin{aligned} m_Y(t) = E(e^{tY}) &= \int_0^\infty e^{ty} f_Y(y)dy \\ &= \int_0^\infty e^{ty} e^{-y}dy \\ &= \int_0^\infty e^{ty-y}dy \\ &= \int_0^\infty e^{-y(1-t)}dy = -\left(\frac{1}{1-t}\right) e^{-y(1-t)}\Bigg|_{y=0}^{\infty}. \end{aligned}$$

$= -\frac{1}{1-t} e^{-(1-t)\times\infty} - \left(-\frac{1}{1-t} e^{-(1-t)\times 0}\right)$

In the last expression, note that

$$\lim_{y\to\infty} e^{-y(1-t)} < \infty$$

if and only if $1 - t > 0$, i.e., $t < 1$. Thus, for $t < 1$, we have

$$m_Y(t) = -\left(\frac{1}{1-t}\right) e^{-y(1-t)}\Bigg|_{y=0}^{\infty} = 0 + \left(\frac{1}{1-t}\right) = \frac{1}{1-t}.$$

$m_Y(t)$ exists at least in $(-1, 1)$

Note that $(-h, h)$ with $h = 1$ is an open neighborhood around zero for which $m_Y(t)$ exists. With the mgf, we can calculate the mean and variance. Differentiating the mgf, we get

$$m_Y'(t) = \frac{d}{dt} m_Y(t) = \frac{d}{dt}\left(\frac{1}{1-t}\right) = \left(\frac{1}{1-t}\right)^2$$

so that

$$E(Y) = \frac{d}{dt} m_Y(t)\bigg|_{t=0} = \left(\frac{1}{1-0}\right)^2 = 1.$$

To find the variance, we first find the second moment. The second derivative of $m_Y(t)$ is

$$\frac{d^2}{dt^2} m_Y(t) = \frac{d}{dt} \underbrace{\left(\frac{1}{1-t}\right)^2}_{m_Y'(t)} = 2\left(\frac{1}{1-t}\right)^3.$$

The second moment is

$$E(Y^2) = \frac{d^2}{dt^2} m_Y(t)\bigg|_{t=0} = 2\left(\frac{1}{1-0}\right)^3 = 2.$$

The computing formula gives

$$V(Y) = E(Y^2) - [E(Y)]^2 = 2 - (1)^2 = 1.$$

EXERCISE: Find $E(Y)$ and $V(Y)$ without using the mgf. □

$$\frac{d}{dt}(1-t)^{-1} = (-1)(1-t)^{-1-1} \times (-1) = (1-t)^{-2}$$

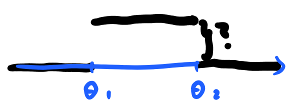
$$\frac{d}{dt}(1-t)^{-2} = -2(1-t)^{-2-1} \times (-1) = 2(1-t)^{-3}$$

$$E(Y) = \int_R y f_Y(y)\,dy = \int_0^\infty y e^{-y}\,dy$$

$$E(Y^2) = \int_R y^2 f_Y(y)\,dy = \int_0^\infty y^2 e^{-y}\,dy$$

4.5 Uniform distribution

TERMINOLOGY: A random variable Y is said to have a **uniform distribution** from θ_1 to θ_2 if its pdf is given by



$$f_Y(y) = \begin{cases} \frac{1}{\theta_2-\theta_1}, & \theta_1 < y < \theta_2 \\ 0, & \text{otherwise.} \end{cases}$$

Shorthand notation is $Y \sim \mathcal{U}(\theta_1, \theta_2)$. Note that this is a valid density because $f_Y(y) > 0$ for all $y \in R = \{y : \theta_1 < y < \theta_2\}$ and

$$\int_{\theta_1}^{\theta_2} f_Y(y)dy = \int_{\theta_1}^{\theta_2} \left(\frac{1}{\theta_2-\theta_1}\right) dy = \frac{y}{\theta_2-\theta_1}\bigg|_{\theta_1}^{\theta_2} = \frac{\theta_2-\theta_1}{\theta_2-\theta_1} = 1.$$

STANDARD UNIFORM: A popular member of the $\mathcal{U}(\theta_1, \theta_2)$ family is the $\mathcal{U}(0, 1)$ distribution; i.e., a uniform distribution with parameters $\theta_1 = 0$ and $\theta_2 = 1$. This model is used extensively in computer programs to simulate random numbers.

$$\int_0^\infty y e^{-y}\,dy \qquad u = y \quad dv = e^{-y}dy \quad v = -e^{-y}$$

$$\int_0^\infty u\,dv = uv\Big|_0^\infty - \int_0^\infty v\,du$$

$$= ye^{-y}\Big|_0^\infty - \int_0^\infty -e^{-y}dy$$

$$= -ye^{-y}\Big|_0^\infty + \int_0^\infty e^{-y}dy$$

$$= \lim_{y\to\infty} \frac{-y}{e^y} - 0 + \int_0^\infty e^{-y}dy$$

$$= 0 - 0 + \int_0^\infty e^{-y}dy$$

$$0 + (-e^{-y})\Big|_0^\infty$$

$$\int_0^\infty y^2 e^{-y}\,dy \qquad u = y^2 \quad dv = e^{-y}dy \quad du = 2y\,dy \quad v = -e^{-y}$$

$$= \int_0^\infty u\,dv = uv\Big|_0^\infty - \int_0^\infty v\,du$$

$$= uv\Big|_0^\infty - \int_0^\infty (-e^{-y})\,2y\,dy$$

$$u\Big|^\infty - \int_0^\infty e^{-y}\,y\,dy$$

$= uv\Big|_0^{\infty} - \int_0^{\infty} (-e^{-y}) \ldots$

$= -y^2 e^{-y}\Big|_0^{\infty} - \int_0^{\infty} e^{-y}\, y\, dy$

$= \lim_{y \to \infty} \frac{-y^2}{e^{y}} + 0 + 2\int_0^{\infty} e^{-y}\, y\, dy$

$= 0 + 0 + 2\int_0^{\infty} y e^{-y}\, dy$

$= 2$

$E(Y^2) = 2$

$= 0 - 0 + \ldots$

$= 0 - 0 + (-e^{-y})\Big|_0^{\infty}$

$= 1$

$V(Y) = 2 - 1^2 = 1$