Section 4.5 Uniform Distribution

Thursday, October 20, 2016 9:40 AM

Section 4.5 Uniform ...

we get

$$m'_Y(t) = \frac{d}{dt} m_Y(t) = \frac{d}{dt}\left(\frac{1}{1-t}\right) = \left(\frac{1}{1-t}\right)^2$$

so that

$$E(Y) = \frac{d}{dt} m_Y(t)\Big|_{t=0} = \left(\frac{1}{1-0}\right)^2 = 1.$$

To find the variance, we first find the second moment. The second derivative of $m_Y(t)$ is

$$\frac{d^2}{dt^2} m_Y(t) = \frac{d}{dt} \underbrace{\left(\frac{1}{1-t}\right)^2}_{m'_Y(t)} = 2\left(\frac{1}{1-t}\right)^3.$$

The second moment is

$$E(Y^2) = \frac{d^2}{dt^2} m_Y(t)\Big|_{t=0} = 2\left(\frac{1}{1-0}\right)^3 = 2.$$

The computing formula gives

$$V(Y) = E(Y^2) - [E(Y)]^2 = 2 - (1)^2 = 1.$$

EXERCISE: Find $E(Y)$ and $V(Y)$ without using the mgf. □

4.5 Uniform distribution

TERMINOLOGY: A random variable Y is said to have a **uniform distribution** from θ_1 to θ_2 if its pdf is given by

$$f_Y(y) = \begin{cases} \frac{1}{\theta_2 - \theta_1}, & \theta_1 < y < \theta_2 \\ 0, & \text{otherwise.} \end{cases}$$

Shorthand notation is $Y \sim \mathcal{U}(\theta_1, \theta_2)$. Note that this is a valid density because $f_Y(y) > 0$ ✓ for all $y \in R = \{y : \theta_1 < y < \theta_2\}$ and

$$\int_{\theta_1}^{\theta_2} f_Y(y)dy = \int_{\theta_1}^{\theta_2} \left(\frac{1}{\theta_2 - \theta_1}\right) dy = \frac{y}{\theta_2 - \theta_1}\Big|_{\theta_1}^{\theta_2} = \frac{\theta_2 - \theta_1}{\theta_2 - \theta_1} = 1. \quad ✓$$

STANDARD UNIFORM: A popular member of the $\mathcal{U}(\theta_1, \theta_2)$ family is the $\mathcal{U}(0, 1)$ distribution; i.e., a uniform distribution with parameters $\theta_1 = 0$ and $\theta_2 = 1$. This model is used extensively in computer programs to simulate random numbers.

Example 4.9. Derive the cdf of $Y \sim \mathcal{U}(\theta_1, \theta_2)$.

SOLUTION. We need to compute $F_Y(y) = P(Y \le y)$ for all $y \in \mathcal{R}$. There are three cases to consider:

pdf $f_Y(y)$, $\frac{1}{\theta_2-\theta_1}$, θ_1, θ_2, y, y, y

- when $y \le \theta_1$,

$$F_Y(y) = \int_{-\infty}^{y} f_Y(t)dt = \int_{-\infty}^{y} 0dt = 0;$$

- when $\theta_1 < y < \theta_2$,

$$F_Y(y) = \int_{-\infty}^{y} f_Y(t)dt = \int_{-\infty}^{\theta_1} 0dt + \int_{\theta_1}^{y} \left(\frac{1}{\theta_2 - \theta_1}\right) dt$$
$$= 0 + \frac{t}{\theta_2 - \theta_1}\Bigg|_{\theta_1}^{y} = \frac{y - \theta_1}{\theta_2 - \theta_1};$$

- when $y \ge \theta_2$,

$$P(Y \le y) = F_Y(y) = \int_{-\infty}^{y} f_Y(t)dt = \int_{-\infty}^{\theta_1} 0dt + \int_{\theta_1}^{\theta_2} \left(\frac{1}{\theta_2 - \theta_1}\right) dt + \int_{\theta_2}^{y} 0dt = 0 + 1 + 0 = 1.$$

Putting this all together, we have

$$F_Y(y) = \begin{cases} 0, & y \le \theta_1 \\ \frac{y-\theta_1}{\theta_2-\theta_1}, & \theta_1 < y < \theta_2 \\ 1, & y \ge \theta_2. \end{cases}$$

The $\mathcal{U}(0,1)$ pdf $f_Y(y)$ and cdf $F_Y(y)$ are plotted side by side in Figure 4.9.

EXERCISE: If $Y \sim \mathcal{U}(0,1)$, find (a) $P(0.2 < Y < 0.4)$ and (b) $P(Y > 0.75)$. □

MEAN AND VARIANCE: If $Y \sim \mathcal{U}(\theta_1, \theta_2)$, then

$$E(Y) = \frac{\theta_1 + \theta_2}{2} \quad \text{and} \quad V(Y) = \frac{(\theta_2 - \theta_1)^2}{12}.$$

UNIFORM MGF: If $Y \sim \mathcal{U}(\theta_1, \theta_2)$, then

$$E[e^{tY}] = m_Y(t) = \begin{cases} \frac{e^{\theta_2 t} - e^{\theta_1 t}}{t(\theta_2 - \theta_1)}, & t \ne 0 \\ 1, & t = 0 \end{cases}$$

EXERCISE: Derive the formulas for $E(Y)$ and $V(Y)$.

$E(Y) = m_Y'(0)$ $\quad E(Y^2) = m_Y''(0)$ $\quad E(Y^k) = m_Y^{(k)}(0)$

$$m_Y(t) = E[e^{tY}] = \int_{-\infty}^{+\infty} e^{ty} f_Y(y)\,dy = \int_{\theta_1}^{\theta_2} e^{ty} \frac{1}{\theta_2-\theta_1} dy$$
$$= \frac{1}{\theta_2-\theta_1} \int_{\theta_1}^{\theta_2} e^{ty} dy = \frac{1}{\theta_2-\theta_1} \left(t^{-1} e^{ty}\right)\Big|_{\theta_1}^{\theta_2} = \frac{e^{t\theta_2} - e^{t\theta_1}}{t(\theta_2-\theta_1)}$$

$t=0$: $m_Y(0) = E[e^0] = E[1] = 1$

$Y \sim U(0,1)$, $\theta_1 = 0$, $\theta_2 = 1$

$$P(0.2 < Y < 0.4) = \int_{0.2}^{0.4} f_Y(t)dt = \int_{0.2}^{0.4} \frac{1}{\theta_2-\theta_1} dt = \int_{0.2}^{0.4} \frac{1}{1-0} dt = \int_{0.2}^{0.4} 1\,dt = t\Big|_{0.2}^{0.4} = 0.2$$

$$f_Y(t) = \begin{cases} 1 & 0<t<1 \\ 0 & \text{o.w.} \end{cases}$$

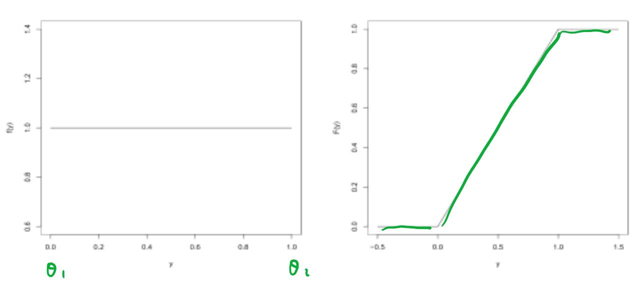
$$P(Y > 0.75) = \int_{0.75}^{+\infty} f_Y(t)dt = \int_{0.75}^{1} f_Y(t)dt + \int_{1}^{+\infty} f_Y(t)dt = \int_{0.75}^{1} 1\,dt + \int_{1}^{+\infty} 0\,dt = t\Big|_{0.75}^{1} + 0 = .25$$

cdf: (a) $P(0.2 < Y < 0.4)$
$$= P(Y < 0.4) - P(Y \le 0.2)$$
$$= P(Y \le 0.4) - P(Y \le 0.2)$$
$$= F_Y(0.4) - F_Y(0.2)$$

$$F_Y(y) = \begin{cases} 0 & y \le 0 \\ y & 0 < y < 1 \\ 1 & y \ge 1 \end{cases} \qquad \theta_1 = 0,\ \theta_2 = 1$$

$$= 0.4 - 0.2 = 0.2$$

(b) $P(Y > 0.75) = 1 - P(Y \le 0.75)$
$$= 1 - F_Y(0.75)$$



pdf, $f_Y(y)$ cdf, $F_Y(y)$

Figure 4.9: *The $\mathcal{U}(0,1)$ probability density function and cumulative distribution function.*

4.6 Normal distribution

TERMINOLOGY: A random variable Y is said to have a **normal distribution** if its pdf is given by

$$f_Y(y) = \begin{cases} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2}, & -\infty < y < \infty \end{cases}$$

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Shorthand notation is $Y \sim \mathcal{N}(\mu, \sigma^2)$. There are two parameters in the normal distribution: the mean $E(Y) = \mu$ and the variance $V(Y) = \sigma^2$.

FACTS:

(a) The $\mathcal{N}(\mu, \sigma^2)$ pdf is symmetric about μ; that is, for any $a \in \mathcal{R}$,

$$f_Y(\mu - a) = f_Y(\mu + a).$$

(b) The $\mathcal{N}(\mu, \sigma^2)$ pdf has points of inflection located at $y = \mu \pm \sigma$ (verify!).

(c) $\lim_{y \to \pm\infty} f_Y(y) = 0$.

$= 1 - F_Y(0.75)$

$= 1 - 0.75 = .25$

Mean and variance of Uniform distribution

Tuesday, October 18, 2016 2:25 PM

$Y \sim U(\theta_1, \theta_2)$ $\quad f_Y(y) = \frac{1}{\theta_2 - \theta_1}$ $\quad \theta_1 < y < \theta_2$

$E(Y) = \frac{\theta_1 + \theta_2}{2}$ $\quad V(Y)$

$$\int_{\theta_1}^{\theta_2} y f_Y(y)\,dy = \int_{\theta_1}^{\theta_2} \frac{y}{\theta_2 - \theta_1}\,dy$$

$$= \frac{1}{\theta_2 - \theta_1} \int_{\theta_1}^{\theta_2} y\,dy$$

$$= \frac{1}{\theta_2 - \theta_1} \left[\left(\frac{1}{2} y^2 \right) \Big|_{\theta_1}^{\theta_2} \right]$$

$a^2 - b^2$

$= (a+b)(a-b)$

$$= \frac{1}{\theta_2 - \theta_1} \times \left(\frac{\theta_2^2}{2} - \frac{\theta_1^2}{2} \right)$$

$$= \frac{1}{2} \times \frac{1}{\theta_2 - \theta_1} \times (\theta_2^2 - \theta_1^2)$$

$$= \frac{1}{2} \frac{1}{\cancel{\theta_2 - \theta_1}} \times \cancel{(\theta_2 - \theta_1)} (\theta_2 + \theta_1)$$

$$= \frac{\theta_2 + \theta_1}{2}$$

$$V(Y) = E(Y^2) - [E(Y)]^2$$

$$E(Y^2) = \int_{\theta_1}^{\theta_2} y^2 f_Y(y)\,dy$$

$$= \int_{\theta_1}^{\theta_2} \frac{y^2}{\theta_2 - \theta_1}\, dy$$

$a^3 - b^3$
$= (a-b)(a^2+ab+b^2)$

$$= \frac{1}{\theta_2-\theta_1} \cdot \frac{1}{3} y^3 \Big|_{\theta_1}^{\theta_2}$$

$$= \frac{1}{\theta_2-\theta_1} \times \frac{1}{3} \times \underline{(\theta_2^3 - \theta_1^3)}$$

$$= \frac{\theta_1^2 + \theta_1\theta_2 + \theta_2^2}{3}$$

$$V(Y) = \frac{\theta_1^2 + \theta_1\theta_2 + \theta_2^2}{3} - \left(\frac{\theta_1+\theta_2}{2}\right)^2$$

$$= \frac{\theta_1^2 + \theta_1\theta_2 + \theta_2^2}{3} - \frac{\theta_1^2 + 2\theta_1\theta_2 + \theta_2^2}{4}$$

$$= \frac{\theta_1^2 - 2\theta_1\theta_2 + \theta_2^2}{12}$$

$$= \frac{(\theta_2 - \theta_1)^2}{12}$$