Section 4.7.1 Exponential distribution

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TERMINOLOGY: A random variable Y is said to have an **exponential distribution** with parameter $\beta > 0$ if its pdf is given by

$$f_Y(y) = \begin{cases} \frac{1}{\beta}e^{-y/\beta}, & y > 0 \\ 0, & \text{otherwise.} \end{cases}$$

$Y \sim \exp(\beta)$

valid pdf: $\begin{cases} 1. \text{ non-neg} \\ 2. \int_{-\infty}^{+\infty} f(y)dy = 1 \end{cases}$

Shorthand notation is $Y \sim \text{exponential}(\beta)$. The value of β determines the scale of the distribution, so it is called a **scale parameter**.

EXERCISE: Show that the exponential pdf integrates to 1.

$$1 = \int_{-\infty}^{+\infty} f_Y(y)dy = \int_0^{\infty} f_Y(y)dy$$
$$= \int_0^{\infty} \frac{1}{\beta} e^{-\frac{y}{\beta}} dy$$
$$= -e^{-\frac{y}{\beta}}\Big|_0^{+\infty}$$
$$= -e^{-\frac{\infty}{\beta}} - (-e^{-\frac{0}{\beta}})$$
$$= 1 - 0 = 0$$

EXPONENTIAL MGF: Suppose that $Y \sim \text{exponential}(\beta)$. The mgf of Y is given by

$$m_Y(t) = \frac{1}{1 - \beta t},$$

for $t < 1/\beta$.

Proof. From the definition of the mgf, we have

$$\begin{aligned} m_Y(t) = E(e^{tY}) = \int_0^{\infty} e^{ty}\left(\frac{1}{\beta}e^{-y/\beta}\right)dy &= \frac{1}{\beta}\int_0^{\infty} e^{ty-y/\beta}dy \\ &= \frac{1}{\beta}\int_0^{\infty} e^{-y[(1/\beta)-t]}dy \\ &= \frac{1}{\beta}\left\{-\left(\frac{1}{\frac{1}{\beta}-t}\right)e^{-y[(1/\beta)-t]}\right\}\Bigg|_{y=0}^{\infty} \\ &= \left(\frac{1}{1-\beta t}\right)\left\{e^{-y[(1/\beta)-t]}\Big|_{y=\infty}^{0}\right\}. \end{aligned}$$

$$\int_{-\infty}^{+\infty} e^{ty} f_Y(y)dy \quad //$$

$$= \frac{1}{1-\beta t}\left\{1 - e^{-\infty\left(\frac{1}{\beta}-t\right)}\right\} < +\infty$$

if $\frac{1}{\beta} - t < 0$

In the last expression, note that

$$\lim_{y\to\infty} e^{-y[(1/\beta)-t]} < \infty$$

if and only if $(1/\beta) - t > 0$, i.e., $t < 1/\beta$. Thus, for $t < 1/\beta$, we have

$$m_Y(t) = \left(\frac{1}{1-\beta t}\right)e^{-y[(1/\beta)-t]}\Big|_{y=\infty}^{0} = \left(\frac{1}{1-\beta t}\right) - 0 = \frac{1}{1-\beta t}.$$

Note that $(-h, h)$ with $h = 1/\beta$ is an open neighborhood around 0 for which $m_Y(t)$ exists. $\square$

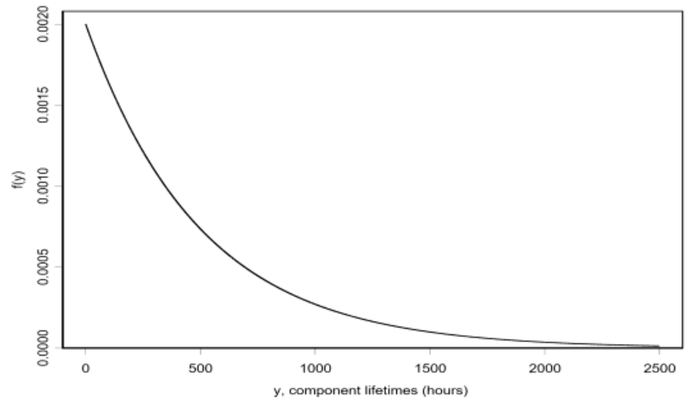


Figure 4.11: *The probability density function, $f_Y(y)$, in Example 4.12. A model for electrical component lifetimes.*

MEAN AND VARIANCE: Suppose that $Y \sim$ exponential(β). The mean and variance of Y are given by

$$E(Y) = \beta \quad \text{and} \quad V(Y) = \beta^2.$$

$M_Y(t) = \frac{1}{1-\beta t} = (1-\beta t)^{-1}$

$E(Y) = M_Y'(0) = (-1)\times(1-\beta t)^{-2}\times(-\beta)\Big|_{t=0}$

$= \beta \times (1-\beta t)^{-2}\Big|_{t=0}$

$= \beta \times (1-0)^{-2} = \beta$

$E(Y^2) = M_Y''(0) = \beta \times (-2)(1-\beta t)^{-3}\times(-\beta)\Big|_{t=0}$

$= 2\beta^2(1-\beta t)^{-3}\Big|_{t=0}$

$= 2\beta^2$

$V(Y) = E(Y^2) - [E(Y)]^2$

$= 2\beta^2 - \beta^2 = \beta^2$

Proof: Exercise. $\square$

Example 4.12. The lifetime of an electrical component has an exponential distribution with mean $\beta = 500$ hours. What is the probability that a randomly selected component fails before 100 hours? lasts between 250 and 750 hours?

SOLUTION. With $\beta = 500$, the pdf for Y is given by

$$f_Y(y) = \begin{cases} \frac{1}{500}e^{-y/500}, & y > 0 \\ 0, & \text{otherwise.} \end{cases}$$

This pdf is depicted in Figure 4.11. Thus, the probability of failing before 100 hours is

$$P(Y < 100) = \int_0^{100} \frac{1}{500}e^{-y/500}dy \approx 0.181.$$

$= \left(-e^{-\frac{y}{500}}\right)\Big|_0^{100}$

Calculated through pdf

$P(Y<100) = F_Y(100) = 1 - e^{-\frac{100}{500}}$

through cdf

$P(250 < Y < 750) = F_Y(750) - F_Y(250)$

$= (1 - e^{-\frac{750}{500}}) - (1 - e^{-\frac{250}{500}})$

Similarly, the probability of failing between 250 and 750 hours is

$$P(250 < Y < 750) = \int_{250}^{750} \frac{1}{500}e^{-y/500}dy \approx 0.383. \square$$

$= -e^{-\frac{y}{500}}\Big|_{250}^{750}$

EXPONENTIAL CDF: Suppose that $Y \sim$ exponential(β). Then, the cdf of Y exists in closed form and is given by

$f_Y(y) = \{ \frac{1}{\beta}e^{-\frac{y}{\beta}} \quad y > 0$

EXPONENTIAL CDF: Suppose that $Y \sim$ exponential(β). Then, the cdf of Y exists in closed form and is given by

$$F_Y(y) = \begin{cases} 0, & y \le 0 \\ 1 - e^{-y/\beta}, & y > 0. \end{cases}$$

$$f_Y(y) = \begin{cases} \frac{1}{\beta} e^{-\frac{y}{\beta}} & y > 0 \\ 0 & \text{o.w.} \end{cases} \Rightarrow Y \text{ can not be negative}$$

$$P(Y \le y) = F_Y(y) = \int_{-\infty}^{y} f(u)\,du$$

when $y \le 0$,

$$F_Y(y) = 0$$

when $y > 0$

$$F_Y(y) = \int_{-\infty}^{y} f(u)\,du = \int_{-\infty}^{0} f(u)\,du + \int_{0}^{y} f(u)\,du = 0 + \int_{0}^{y} \frac{1}{\beta} e^{-\frac{u}{\beta}}\,du = \left(-e^{-\frac{u}{\beta}}\right)\Big|_{0}^{y} = 1 - e^{-\frac{y}{\beta}}$$

Proof. Exercise. □

THE MEMORYLESS PROPERTY: Suppose that $Y \sim$ exponential(β), and let r and s be positive constants. Then

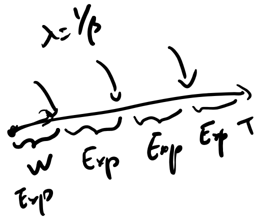
$$\frac{P(Y>r+s)}{P(Y>r)} = e^{-(r+s)/\beta} / e^{-r/\beta} = e^{-s/\beta}$$

$$\frac{P(Y>r+s \cap Y>s)}{P(Y>r)} = P(Y > r + s | Y > r) = P(Y > s). = e^{-\frac{s}{\beta}}$$

That is, given that the lifetime Y has exceeded r, the probability that Y exceeds $r+s$ (i.e., an additional s units) is the same as if we were to look at Y unconditionally lasting until time s. Put another way, that Y has actually "made it" to time r has been forgotten. The exponential random variable is the only continuous random variable that possesses the memoryless property. $P(Y>y) = 1 - P(Y \le y) = e^{-\frac{y}{\beta}}$

RELATIONSHIP WITH A POISSON PROCESS: Suppose that we are observing events according to a Poisson process with rate $\lambda = 1/\beta$, and let the random variable W denote the time until the first occurrence. Then, $W \sim$ exponential(β).

Proof: Clearly, W is a continuous random variable with nonnegative support. Thus, for $w \ge 0$, we have



$$\begin{aligned} F_W(w) = P(W \le w) &= 1 - P(W > w) \\ &= 1 - P(\{\text{no events in } [0, w]\}) \\ &= 1 - \frac{e^{-\lambda w}(\lambda w)^0}{0!} \\ &= 1 - e^{-\lambda w}. \end{aligned}$$

Denote # of events in $[0, w]$ by Y

$Y \sim$ Poisson (λw)

$P(Y=0)$

↳ cdf of exp$(\frac{1}{\lambda} = \beta)$

Substituting $\lambda = 1/\beta$, we have $F_W(w) = 1 - e^{-w/\beta}$, the cdf of an exponential random variable with mean β. Thus, the result follows. □

the problem tells you it is exp by using Poisson Process

Example 4.13. Suppose that customers arrive at a check-out according to a Poisson process with mean $\lambda = 12$ per hour. What is the probability that we will have to wait longer than 10 minutes to see the first customer? NOTE: 10 minutes is 1/6th of an hour.

SOLUTION. The time until the first arrival, say W, follows an exponential distribution with mean $\beta = 1/\lambda = 1/12$, so that the cdf of W, for $w > 0$, is $F_W(w) = 1 - e^{-12w}$. Thus, the desired probability is

$W \sim \exp(\frac{1}{12})$ $\quad P(W > 10 \text{ min}) = P(W > \frac{1}{6} \text{ hour})$

$$P(W > 1/6) = 1 - P(W \leq 1/6) = 1 - F_W(1/6) = 1 - [1 - e^{-12(1/6)}] = e^{-2} \approx 0.135. \ \square$$

4.7.2 Gamma distribution

TERMINOLOGY: The **gamma function** is a real function of t, defined by

$$\Gamma(t) = \int_0^\infty y^{t-1} e^{-y} dy,$$

$\Gamma(1) = \int_0^\infty y^{1-1} e^{-y} dy$
$= \int_0^\infty e^{-y} dy = (-e^{-y})\Big|_0^\infty$
$= 1$

for all $t > 0$. The gamma function satisfies the recursive relationship

$$\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1),$$

for $\alpha > 1$. From this fact, we can deduce that if α is an integer, then

$$\Gamma(\alpha) = (\alpha - 1)!$$

$\Gamma(2) = (2-1)\Gamma(2-1)$
$= 1 \times \Gamma(1) = 1$
$\Gamma(3) = (3-1) \times \Gamma(3-1)$
$= 2 \times \Gamma(2) = 2$
$\vdots$
$\Gamma(n) = (n-1)!$

For example, $\Gamma(5) = 4! = 24$.

TERMINOLOGY: A random variable Y is said to have a **gamma distribution** with parameters $\alpha > 0$ and $\beta > 0$ if its pdf is given by

$$f_Y(y) = \begin{cases} \frac{1}{\Gamma(\alpha)\beta^\alpha} y^{\alpha-1} e^{-y/\beta}, & y > 0 \\ 0, & \text{otherwise.} \end{cases}$$

Shorthand notation is $Y \sim \text{gamma}(\alpha, \beta)$. The gamma distribution is indexed by two parameters:

$$\alpha = \text{the } \textbf{shape} \text{ parameter}$$
$$\beta = \text{the } \textbf{scale} \text{ parameter.}$$