Section 4.7.2 Gamma distribution

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Section 4.7.2 Ga...

Example 4.13. Suppose that customers arrive at a check-out according to a Poisson process with mean $\lambda = 12$ per hour. What is the probability that we will have to wait longer than 10 minutes to see the first customer? NOTE: 10 minutes is 1/6th of an hour.

SOLUTION. The time until the first arrival, say W, follows an exponential distribution with mean $\beta = 1/\lambda = 1/12$, so that the cdf of W, for $w > 0$, is $F_W(w) = 1 - e^{-12w}$. Thus, the desired probability is

$$P(W > 1/6) = 1 - P(W \le 1/6) = 1 - F_W(1/6) = 1 - [1 - e^{-12(1/6)}] = e^{-2} \approx 0.135. \quad \square$$

$Y \sim \text{Exponential}(\beta) = \text{Gamma}(\alpha = 1, \beta)$

$$f_Y(y) = \begin{cases} \frac{1}{\beta} e^{-\frac{y}{\beta}} & y > 0 \\ 0 & \text{otherwise} \end{cases}$$

4.7.2 Gamma distribution

TERMINOLOGY: The **gamma function** is a real function of t, defined by

$$\Gamma(t) = \int_0^\infty y^{t-1} e^{-y} dy,$$

$\Gamma(1) = \int_0^\infty y^{1-1} e^{-y} dy = \int_0^\infty e^{-y} dy = 1$

for all $t > 0$. The gamma function satisfies the recursive relationship

$$\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1),$$

for $\alpha > 1$. From this fact, we can deduce that if α is an integer, then

$$\Gamma(\alpha) = (\alpha - 1)!$$

For example, $\Gamma(5) = 4! = 24$.

$Y \sim \text{Gamma}(\alpha, \beta)$

pdf:

$$f_Y(y) = \begin{cases} \underbrace{\frac{1}{\Gamma(\alpha)} \frac{1}{\beta^\alpha}}_{\text{Constant}} \times \underbrace{y^{\alpha-1} e^{-\frac{y}{\beta}}}_{\text{Kernel}} & y > 0 \\ 0 & \text{o.w.} \end{cases}$$

Valid? (1) $f_Y(y) \ge 0$ ✓

(2) $\int_{-\infty}^{+\infty} f_Y(y) dy = 1$?

TERMINOLOGY: A random variable Y is said to have a gamma distribution with parameters $\alpha > 0$ and $\beta > 0$ if its pdf is given by

$$f_Y(y) = \begin{cases} \frac{1}{\Gamma(\alpha)\beta^\alpha} y^{\alpha-1} e^{-y/\beta}, & y > 0 \\ 0, & \text{otherwise.} \end{cases}$$

Shorthand notation is $Y \sim \text{gamma}(\alpha, \beta)$. The gamma distribution is indexed by two parameters:

$$\begin{aligned} \alpha &= \text{the shape parameter} \\ \beta &= \text{the scale parameter.} \end{aligned}$$

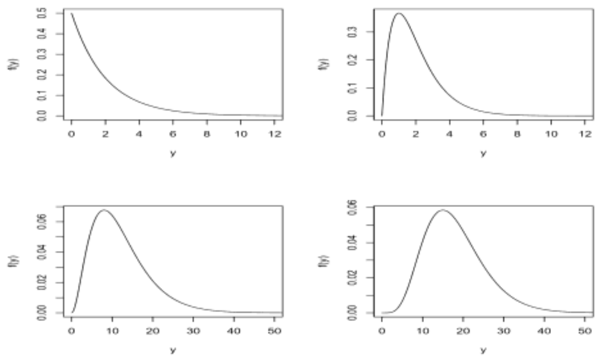

Figure 4.12: *Four gamma pdfs. Upper left:* $\alpha = 1$, $\beta = 2$. *Upper right:* $\alpha = 2$, $\beta = 1$. *Lower left:* $\alpha = 3$, $\beta = 4$. *Lower right:* $\alpha = 6$, $\beta = 3$.

REMARK: By changing the values of α and β, the gamma pdf can assume many shapes. This makes the gamma distribution popular for modeling lifetime data. Note that when $\alpha = 1$, the gamma pdf reduces to the exponential(β) pdf. That is, the exponential pdf is a "special" gamma pdf.

Example 4.14. Show that the gamma(α, β) pdf integrates to 1.

$\int_{-\infty}^{+\infty} f_Y(y)dy = 1$ $\quad f_Y(y)=0,\ y<0$

SOLUTION. Change the variable of integration to $u = y/\beta$ so that $du = dy/\beta$ and $dy = \beta du$. We have

$$\begin{aligned} 1 = \int_0^\infty f_Y(y)dy &= \int_0^\infty \frac{1}{\Gamma(\alpha)\beta^\alpha} y^{\alpha-1}e^{-y/\beta}dy \\ &= \frac{1}{\Gamma(\alpha)}\int_0^\infty \frac{1}{\beta^\alpha}(\beta u)^{\alpha-1}e^{-\beta u/\beta}\beta du \\ &= \frac{1}{\Gamma(\alpha)}\int_0^\infty u^{\alpha-1}e^{-u}du \\ &= \frac{\Gamma(\alpha)}{\Gamma(\alpha)} = 1. \ \square \end{aligned}$$

$u = \frac{y}{\beta}$ $\quad y = u \times \beta$

$du = \frac{1}{\beta}dy$ $\quad dy = \beta \cdot du$

$\Gamma(t) = \int_0^\infty u^{t-1}e^{-u}du$

$\int_0^\infty \frac{1}{\Gamma(\alpha)}\frac{1}{\beta^\alpha} y^{\alpha-1}e^{-\frac{y}{\beta}}dy = 1$

$\Rightarrow \Gamma(\alpha)\beta^\alpha = \int_0^\infty y^{\alpha-1}e^{-\frac{y}{\beta}}dy$!!!

$\frac{1}{\beta^\alpha} y^{\alpha-1} e^{-y/\beta} dy$

$= \frac{1}{\cancel{\beta^\alpha}}(u\beta)^{\alpha-1} e^{-\frac{u\cancel{\beta}}{\cancel{\beta}}} \cancel{\beta} du = u^{\alpha-1}e^{-u}du$

$\int_0^\infty y e^{-y}dy = \int_0^\infty y^{2-1}e^{-\frac{y}{1}}dy$

$= \Gamma(2)\times 1^2$

$= 1! \times 1 = 1$

$\int_0^\infty y^3 e^{-y}dy$

$= \Gamma(4) \times 1^4 = 3! = 6$

mgf: $m_Y(t) = \left(\frac{1}{1-\beta t}\right)^\alpha$

for $t < \frac{1}{\beta}$

GAMMA MGF: Suppose that $Y \sim$ gamma(α, β). The mgf of Y is

$$m_Y(t) = \left(\frac{1}{1-\beta t}\right)^\alpha,$$

for $t < 1/\beta$.

Proof. From the definition of the mgf, we have

$$\begin{aligned} m_Y(t) = E(e^{tY}) &= \int_0^\infty e^{ty}\left[\frac{1}{\Gamma(\alpha)\beta^\alpha}y^{\alpha-1}e^{-y/\beta}\right]dy \\ &= \int_0^\infty \frac{1}{\Gamma(\alpha)\beta^\alpha}y^{\alpha-1}e^{ty-y/\beta}dy \\ &= \int_0^\infty \frac{1}{\Gamma(\alpha)\beta^\alpha}y^{\alpha-1}e^{-y[(1/\beta)-t]}dy \\ &= \int_0^\infty \frac{1}{\Gamma(\alpha)\beta^\alpha}y^{\alpha-1}e^{-y/[(1/\beta)-t]^{-1}}dy \\ &= \frac{\eta^\alpha}{\beta^\alpha}\int_0^\infty \frac{1}{\Gamma(\alpha)\eta^\alpha}y^{\alpha-1}e^{-y/\eta}dy, \end{aligned}$$

where $\eta = [(1/\beta) - t]^{-1}$. If $\eta > 0 \iff t < 1/\beta$, then the last integral equals 1, because

$$= \frac{\eta^\alpha}{\beta^\alpha} \int_0^\infty \frac{1}{\Gamma(\alpha)\eta^\alpha} y^{\alpha-1} e^{-y/\eta} dy,$$

where $\eta = [(1/\beta) - t]^{-1}$. If $\eta > 0 \iff t < 1/\beta$, then the last integral equals 1, because the integrand is the gamma(α, η) pdf and integration is over $R = \{y : 0 < y < \infty\}$. Thus,

$$m_Y(t) = \left(\frac{\eta}{\beta}\right)^\alpha = \left\{\frac{1}{\beta[(1/\beta) - t]}\right\}^\alpha = \left(\frac{1}{1 - \beta t}\right)^\alpha.$$

Note that $(-h, h)$ with $h = 1/\beta$ is an open neighborhood around 0 for which $m_Y(t)$ exists. □

$E(Y) = \alpha\beta$

$V(Y) = \alpha\beta^2$

MEAN AND VARIANCE: If $Y \sim$ gamma(α, β), then

$$E(Y) = \alpha\beta \quad \text{and} \quad V(Y) = \alpha\beta^2.$$

NOTE: Upon closer inspection, we see that the nonzero part of the gamma(α, β) pdf

$$f_Y(y) = \frac{1}{\Gamma(\alpha)\beta^\alpha} y^{\alpha-1} e^{-y/\beta}$$

consists of two parts:

- the kernel of the pdf: $y^{\alpha-1} e^{-y/\beta}$
- a constant out front: $1/\Gamma(\alpha)\beta^\alpha$.

The kernel is the "guts" of the formula, while the constant out front is simply the "right quantity" that makes $f_Y(y)$ a valid pdf; i.e., the constant which makes $f_Y(y)$ integrate to 1. Note that because

$$\int_0^\infty \frac{1}{\Gamma(\alpha)\beta^\alpha} y^{\alpha-1} e^{-y/\beta} dy = 1,$$

it follows immediately that

$$\int_0^\infty y^{\alpha-1} e^{-y/\beta} dy = \Gamma(\alpha)\beta^\alpha.$$

This fact is extremely fascinating in its own right, and it is very helpful too; we will use it repeatedly.

$y^2 e^{-\frac{y}{4}}$ $\quad \alpha - 1 = 2 \Rightarrow \alpha = 3$, $\beta = 4$

Example 4.15. Suppose that Y has pdf given by

$$f_Y(y) = \begin{cases} cy^2 e^{-y/4}, & y > 0 \\ 0, & \text{otherwise.} \end{cases}$$

1. $f_Y(y) \geq 0 \Rightarrow c \geq 0$
2. $\int_{-\infty}^{+\infty} f_Y(y)dy = 1$

(a) What is the value of c that makes this a valid pdf?

(b) What is the mgf of Y?

(c) What are the mean and variance of Y?

Solve $\int_0^\infty f_Y(y)dy = 1$ for c

SOLUTIONS. Note that $y^2 e^{-y/4}$ is a gamma kernel with $\alpha = 3$ and $\beta = 4$. Thus, the constant out front is

$Y \sim \text{Gamma}(\alpha=3, \beta=4)$

$$c = \frac{1}{\Gamma(\alpha)\beta^\alpha} = \frac{1}{\Gamma(3)4^3} = \frac{1}{2(64)} = \frac{1}{128}.$$

$$\int_0^\infty c\, y^2 e^{-\frac{y}{4}} dy = 1$$

$$c \times \underbrace{\int_0^\infty y^2 e^{-\frac{y}{4}} dy}_{} = 1$$

$$c \times \Gamma(3) 4^3 = 1$$

The mgf of Y is

$t < \frac{1}{\beta}$

$$m_Y(t) = \left(\frac{1}{1-\beta t}\right)^\alpha = \left(\frac{1}{1-4t}\right)^3,$$

for $t < 1/4$. Finally,

$$E(Y) = \alpha\beta = 3(4) = 12$$
$$V(Y) = \alpha\beta^2 = 3(4^2) = 48.$$

(d) $P(1 < Y < 3) = \int_1^3 \frac{1}{128} y^2 e^{-\frac{y}{4}} dy$

TI-84

RELATIONSHIP WITH A POISSON PROCESS: Suppose that we are observing events according to a Poisson process with rate $\lambda = 1/\beta$, and let the random variable W denote the time until the αth occurrence. Then, $W \sim \text{gamma}(\alpha, \beta)$.

$\beta = \frac{1}{\lambda}$

Proof: Clearly, W is a continuous random variable with nonnegative support. Thus, for $w \geq 0$, we have

W: Time till the α th occurrence.

$$\begin{aligned} F_W(w) = P(W \leq w) &= 1 - P(W > w) \\ &= 1 - P(\{\text{fewer than } \alpha \text{ events in } [0, w]\}) \\ &= 1 - \sum_{j=0}^{\alpha-1} \frac{e^{-\lambda w}(\lambda w)^j}{j!}. \end{aligned}$$

The pdf of W, $f_W(w)$, is equal to $F'_W(w)$, provided that this derivative exists. For $w > 0$,

$$\begin{aligned} f_W(w) = F'_W(w) &= \lambda e^{-\lambda w} - e^{-\lambda w} \underbrace{\sum_{j=1}^{\alpha-1} \left[\frac{j(\lambda w)^{j-1}\lambda}{j!} - \frac{(\lambda w)^j \lambda}{j!} \right]}_{\text{telescoping sum}} \\ &= \lambda e^{-\lambda w} - e^{-\lambda w} \left[\lambda - \frac{\lambda(\lambda w)^{\alpha-1}}{(\alpha-1)!} \right] \\ &= \frac{\lambda(\lambda w)^{\alpha-1} e^{-\lambda w}}{(\alpha-1)!} = \frac{\lambda^\alpha}{\Gamma(\alpha)} w^{\alpha-1} e^{-\lambda w}. \end{aligned}$$

$w^{\alpha-1} e^{-\lambda w}$

$w^{\alpha-1} e^{-\frac{w}{\beta}}$ match

$\boxed{\alpha = \alpha, \ \beta = \frac{1}{\lambda}}$

Substituting $\lambda = 1/\beta$,

$$f_W(w) = \frac{1}{\Gamma(\alpha)\beta^\alpha} w^{\alpha-1} e^{-w/\beta},$$

for $w > 0$, which is the pdf for the gamma(α, β) distribution. $\square$

Example 4.16. Suppose that customers arrive at a check-out according to a Poisson process with mean $\lambda = 12$ per hour. What is the probability that we will have to wait longer than 10 minutes to see the third customer? NOTE: 10 minutes is 1/6th of an hour.

SOLUTION. The time until the third arrival, say W, follows a gamma distribution with parameters $\alpha = 3$ and $\beta = 1/\lambda = 1/12$, so that the pdf of W, for $w > 0$,

$$f_W(w) = 864 w^2 e^{-12w}.$$

$W \sim \text{Gamma}(\alpha=3, \beta=\frac{1}{12})$

Thus, the desired probability is

density function

$$\begin{aligned} P(W > 1/6) &= 1 - P(W \leq 1/6) \\ &= 1 - \int_0^{1/6} 864 w^2 e^{-12w} dw \approx 0.677. \ \square \end{aligned}$$

$= 1 - \int_0^{1/6} 864 y^2 e^{-12y} dy$

 TI-84

1. pdf
2. $\Gamma(\alpha)\beta^\alpha =$ ______ .
3. MGF
4. $E(Y)$. $V(Y)$
5. Relationship with Poisson Process
6. Calculate Probability (TI-84)