Section 4.9 Chebyshev's Inequality

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Section 4.9 Chebyshe...

infection proportion is above 10 percent" (i.e., a "success"). Because days are assumed independent, $X \sim b(5, 0.122)$ and

$$\begin{aligned} P(X \geq 2) &= 1 - P(X=0) - P(X=1) \\ &= 1 - \binom{5}{0}(0.122)^0(1-0.122)^5 - \binom{5}{1}(0.122)^1(1-0.122)^4 \approx 0.116. \ \square \end{aligned}$$

4.9 Chebyshev's Inequality

MARKOV'S INEQUALITY: Suppose that X is a nonnegative random variable with pdf (pmf) $f_X(x)$ and let c be a positive constant. Markov's Inequality puts a bound on the upper tail probability $P(X > c)$; that is,

$$P(X > c) \leq \frac{E(X)}{c}.$$

Proof. First, define the event $B = \{x : x > c\}$. We know that

$$= \int_c^{\infty} x f_x(x)dx \quad c>0$$

$$\begin{aligned} E(X) = \int_0^{\infty} x f_X(x)dx &= \underbrace{\int_B x f_X(x)dx}_{= \int_c^{\infty} x f_x(x)dx} + \underbrace{\int_{\overline{B}} x f_X(x)dx}_{\geq 0} \\ &\geq \int_B x f_X(x)dx = \int_c^{+\infty} x f_x(x)dx \\ &\geq \int_B c f_X(x)dx = cP(X > c). \ \square \end{aligned}$$

when $x \geq c$

$x f_x(x) \geq c f_x(x)$

$\int_B c f_X(x)dx = c \int_c^{+\infty} f_X(x)dx$

CHEBYSHEV'S INEQUALITY: Let Y be any random variable, discrete or continuous, with mean μ and variance $\sigma^2 < \infty$. For $k > 0$,

$$P(|Y - \mu| > k\sigma) \leq \frac{1}{k^2}.$$

$P(X>c) \leq \frac{E(X)}{c}$

$P((Y-\mu)^2 > k^2\sigma^2) \leq \frac{E[(Y-\mu)^2]}{k^2\sigma^2}$

Proof. Applying Markov's Inequality with $X = (Y-\mu)^2$ and $c = k^2\sigma^2$, we have

$$P(|Y-\mu| > k\sigma) = P[(Y-\mu)^2 > k^2\sigma^2] \leq \frac{E[(Y-\mu)^2]}{k^2\sigma^2} = \frac{\sigma^2}{k^2\sigma^2} = \frac{1}{k^2}. \ \square$$

$E[(Y-\mu)^2]$

$= V(Y)$

$$P(|Y-\mu|>k\sigma) = P[(Y-\mu)^2 > k^2\sigma^2] \le \frac{E[(Y-\mu)^2]}{k^2\sigma^2} = \frac{\sigma^2}{k^2\sigma^2} = \frac{1}{k^2}. \square$$

$E[(Y-\mu)^2]$
$= V(Y)$
$= EY^2 - \mu^2$

REMARK: The beauty of Chebyshev's result is that it applies to any random variable Y. In words, $P(|Y-\mu|>k\sigma)$ is the probability that the random variable Y will differ from the mean μ by more than k standard deviations. If we do not know how Y is distributed,

we can not compute $P(|Y-\mu|>k\sigma)$ exactly, but, at least we can put an upper bound on this probability; this is what Chebyshev's result allows us to do. Note that

$$P(|Y-\mu|>k\sigma) = 1 - P(|Y-\mu|\le k\sigma) = 1 - P(\mu - k\sigma \le Y \le \mu + k\sigma).$$

Thus, it must be the case that

$1 - P(|Y-\mu|>k\sigma) \ge 1 - \frac{1}{k^2}$

$$P(|Y-\mu|\le k\sigma) = P(\mu - k\sigma \le Y \le \mu + k\sigma) \ge 1 - \frac{1}{k^2}.$$

Example 4.18. Suppose that Y represents the amount of precipitation (in inches) observed annually in Barrow, AK. The exact probability distribution for Y is unknown, but, from historical information, it is posited that $\mu = 4.5$ and $\sigma = 1$. What is a lower bound on the probability that there will be between 2.5 and 6.5 inches of precipitation during the next year?

$k=2$

$2.5 = 4.5-2 = \mu-2\sigma$, $6.5 = 4.5+2 = \mu+2\sigma$

SOLUTION: We want to compute a lower bound for $P(2.5 \le Y \le 6.5)$. Note that

$$P(2.5 \le Y \le 6.5) = P(|Y-\mu|\le 2\sigma) \ge 1 - \frac{1}{2^2} = 0.75.$$

$P(\mu-2\sigma \le Y \le \mu+2\sigma)$

Thus, we know that $P(2.5 \le Y \le 6.5) \ge 0.75$. The chances are good that the annual precipitation will be between 2.5 and 6.5 inches.

4.10 Expectations of piecewise functions and mixed distributions

4.10.1 Expectations of piecewise functions

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4.10.1 Expectations of piecewise functions

RECALL: Suppose that Y is a continuous random variable with pdf $f_Y(y)$ and support R. Let $g(Y)$ be a function of Y. The **expected value** of $g(Y)$ is given by

$$E[g(Y)] = \int_R g(y) f_Y(y) dy,$$

provided that this integral exists.

REMARK: In mathematical expectation examples up until now, we have always considered functions g which were continuous and differentiable everywhere; e.g., $g(y) = y^2$,