Section 5.3

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Example 5.1. Tornados are natural disasters that cause millions of dollars in damage each year. An actuary determines that the annual numbers of tornadoes in two Iowa counties (Lee and Van Buren) are jointly distributed as indicated in the table below. Let Y_1 and Y_2 denote the number of tornados seen each year in Lee and Van Buren counties, respectively.

$p_{Y_1,Y_2}(y_1,y_2)$	$y_2=0$	$y_2=1$	$y_2=2$	$y_2=3$
$y_1=0$	0.12	0.06	0.05	0.02
$y_1=1$	0.13	0.15	0.12	0.03
$y_1=2$	0.05	0.15	0.10	0.02

(a) What is the probability that there is no more than one tornado seen in the two counties combined?

SOLUTION. We want to compute $P(Y_1+Y_2\leq 1)$. Note that the support points which correspond to the event $\{Y_1+Y_2\leq 1\}$ are $(0,0),(0,1)$ and $(1,0)$. Thus,

$$\begin{aligned} P(Y_1+Y_2\leq 1) &= p_{Y_1,Y_2}(0,0)+p_{Y_1,Y_2}(1,0)+p_{Y_1,Y_2}(0,1) \\ &= 0.12+0.13+0.06=0.31. \end{aligned}$$

(b) What is the probability that there are two tornadoes in Lee County?

SOLUTION. We want to compute $P(Y_1=2)$. Note that the support points which

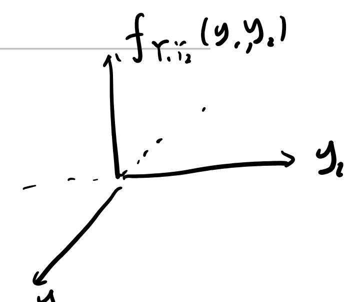
(b) What is the probability that there are two tornadoes in Lee County?

SOLUTION. We want to compute $P(Y_1 = 2)$. Note that the support points which correspond to the event $\{Y_1 = 2\}$ are $(2,0), (2,1), (2,2)$ and $(2,3)$. Thus,

$$\begin{aligned} P(Y_1 = 2) &= p_{Y_1,Y_2}(2,0) + p_{Y_1,Y_2}(2,1) + p_{Y_1,Y_2}(2,2) + p_{Y_1,Y_2}(2,3) \\ &= 0.05 + 0.15 + 0.10 + 0.02 = 0.32. \ \square \end{aligned}$$

5.3 Continuous random vectors

TERMINOLOGY: Let Y_1 and Y_2 be continuous random variables. Then, (Y_1, Y_2) is called a **continuous random vector**, and the **joint probability density function (pdf)** of Y_1 and Y_2 is denoted by $f_{Y_1,Y_2}(y_1, y_2)$. The joint pdf $f_{Y_1,Y_2}(y_1, y_2)$ is a three-dimensional function whose domain is R, the two-dimensional support of (Y_1, Y_2).



$R = \{(y_1, y_2): f_{Y_1,Y_2}(y_1,y_2) > 0\}$

PROPERTIES: The function $f_{Y_1,Y_2}(y_1, y_2)$ has the following properties:

(1) $f_{Y_1,Y_2}(y_1, y_2) > 0$, for all $(y_1, y_2) \in R$ ① $f_{Y_1,Y_2}(y_1,y_2) \geq 0$

(2) The function $f_{Y_1,Y_2}(y_1, y_2)$ integrates to 1 over its support R; i.e.,

$$\int_R f_{Y_1,Y_2}(y_1, y_2)dy_1dy_2 = 1.$$

② $\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_{Y_1,Y_2}(y_1,y_2)dy_1\,dy_2 = 1$

We realize this is a double integral since R is a two-dimensional set.

RESULT: Suppose that (Y_1, Y_2) is a continuous random vector with joint pdf $f_{Y_1,Y_2}(y_1, y_2)$. Then,

$$P[(Y_1, Y_2) \in B] = \int_B f_{Y_1,Y_2}(y_1, y_2)dy_1dy_2,$$

for any set $B \subset \mathcal{R}^2$. We realize that this is a double integral since B is a two-dimensional set in the (y_1, y_2) plane. Therefore, geometrically, $P[(Y_1, Y_2) \in B]$ is the volume under the three-dimensional function $f_{Y_1,Y_2}(y_1, y_2)$ over the two-dimensional set B.

TERMINOLOGY: Suppose that (Y_1, Y_2) is a continuous random vector with joint pdf the three-dimensional function $f_{Y_1,Y_2}(y_1, y_2)$ over the two-dimensional set B.

TERMINOLOGY: Suppose that (Y_1, Y_2) is a continuous random vector with joint pdf $f_{Y_1,Y_2}(y_1, y_2)$. The **joint cumulative distribution function (cdf)** for (Y_1, Y_2) is given by

$$\int_{-\infty}^{y_1}\int_{-\infty}^{y_2} f_{Y_1,Y_2}(t_1,t_2)dt_1dt_2 = F_{Y_1,Y_2}(y_1, y_2) \equiv P(Y_1 \leq y_1, Y_2 \leq y_2) = \int_{-\infty}^{y_2}\int_{-\infty}^{y_1} f_{Y_1,Y_2}(t_1, t_2)dt_1dt_2,$$

← how to find cdf based on pdf

for all $(y_1, y_2) \in \mathcal{R}^2$. It follows upon differentiation that the joint pdf is given by

$$\frac{\partial^2}{\partial y_2 \partial y_1} F_{Y_1,Y_2}(y_1,y_2) = f_{Y_1,Y_2}(y_1, y_2) = \frac{\partial^2}{\partial y_1 \partial y_2} F_{Y_1,Y_2}(y_1, y_2),$$

← how to find pdf based on cdf

wherever these mixed partial derivatives are defined.

Example 5.2. A bank operates with a drive-up facility and a walk-up window. On a randomly selected day, let

$$\begin{aligned} Y_1 &= \text{proportion of time the drive-up facility is in use} \\ Y_2 &= \text{proportion of time the walk-up facility is in use.} \end{aligned}$$

Suppose that the joint pdf of (Y_1, Y_2) is given by

$$f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} \frac{6}{5}(y_1 + y_2^2), & 0 < y_1 < 1, 0 < y_2 < 1 \\ 0, & \text{otherwise.} \end{cases}$$

$R = \{(y_1, y_2): 0<y_1<1, 0<y_2<1\}$

Note that the support in this example is

$$R = \{(y_1, y_2) : 0 < y_1 < 1,\ 0 < y_2 < 1\}.$$

It is very helpful to plot the support of (Y_1, Y_2) **in the** (y_1, y_2) **plane.**

(a) What is the probability that neither facility is busy more than 1/4 of the day? That is, what is $P(Y_1 \leq 1/4, Y_2 \leq 1/4)$?

SOLUTION. Here, we want to integrate the joint pdf $f_{Y_1,Y_2}(y_1, y_2)$ over the set

$$B = \{(y_1, y_2) : 0 < y_1 \leq 1/4,\ 0 < y_2 \leq 1/4\}.$$

The desired probability is $= \int_B f_{Y_1,Y_2}(y_1,y_2)dy_1dy_2$

The desired probability is $= \int_B f_{Y_1,Y_2}(y_1,y_2)\,dy_1\,dy_2$

$(y_1 + y_2^2)$

$y_1 y_2 + \frac{1}{3} y_2^3$

$$
\begin{aligned}
P(Y_1 \le 1/4, Y_2 \le 1/4) &= \int_{y_1=0}^{1/4}\int_{y_2=0}^{1/4} \frac{6}{5}(y_1+y_2^2)\,dy_2\,dy_1 \\
&= \frac{6}{5}\int_{y_1=0}^{1/4}\left[\left(y_1y_2+\frac{y_2^3}{3}\right)\Bigg|_{y_2=0}^{1/4}\right]dy_1 \\
&= \frac{6}{5}\int_{y_1=0}^{1/4}\left(\frac{y_1}{4}+\frac{1}{192}\right)dy_1 \\
&= \frac{6}{5}\left[\left(\frac{y_1^2}{8}+\frac{y_1}{192}\right)\Bigg|_{y_1=0}^{1/4}\right] = \frac{6}{5}\left(\frac{1}{128}+\frac{1}{768}\right) \approx 0.0109.
\end{aligned}
$$

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(b) Find the probability that the proportion of time the drive-up facility is in use is less than the proportion of time the walk-up facility is in use; i.e., compute $P(Y_1 < Y_2)$.

Solution. Here, we want to integrate the joint pdf $f_{Y_1,Y_2}(y_1,y_2)$ over the set

1st step:

$$B = \{(y_1,y_2) : 0 < y_1 < y_2 < 1\}.$$

The desired probability is $= \int_B f_{Y_1,Y_2}(y_1,y_2)\,dy_1\,dy_2$

$y_1 + y_2^2$

const

$$
\begin{aligned}
P(Y_1 < Y_2) &= \int_{y_2=0}^{1}\int_{y_1=0}^{y_2} \frac{6}{5}(y_1+y_2^2)\,dy_1\,dy_2 \\
&= \frac{6}{5}\int_{y_2=0}^{1}\left[\left(\frac{y_1^2}{2}+y_1y_2^2\right)\Bigg|_{y_1=0}^{y_2}\right]dy_2 \\
&= \frac{6}{5}\int_{y_2=0}^{1}\left(\frac{y_2^2}{2}+y_2^3\right)dy_2 \\
&= \frac{6}{5}\left[\left(\frac{y_2^3}{6}+\frac{y_2^4}{4}\right)\Bigg|_{y_2=0}^{1}\right] = \frac{6}{5}\left(\frac{1}{6}+\frac{1}{4}\right) = 0.5. \ \square
\end{aligned}
$$

2nd step

$\frac{1}{2} y_1^2 + y_1 y_2^2$

3nd step