Section 5.4 Marginal Distribution

Thursday, November 10, 2016 1:01 PM

Section 5.4 Marginal ...

5.4 Marginal distributions

DISCRETE CASE: The joint pmf of (Y_1, Y_2) in Example 5.1 is depicted below (in the inner rectangular part of the table). The marginal distributions of Y_1 and Y_2 are catalogued in the margins of the table.

$p_{Y_1,Y_2}(y_1, y_2)$	$y_2 = 0$	$y_2 = 1$	$y_2 = 2$	$y_2 = 3$	$p_{Y_1}(y_1)$
$y_1 = 0$	0.12	0.06	0.05	0.02	0.25
$y_1 = 1$	0.13	0.15	0.12	0.03	0.43
$y_1 = 2$	0.05	0.15	0.10	0.02	0.32
$p_{Y_2}(y_2)$	0.30	0.36	0.27	0.07	1

$P(Y_1 = 0)$

TERMINOLOGY: Let (Y_1, Y_2) be a **discrete** random vector with pmf $p_{Y_1,Y_2}(y_1, y_2)$. The **marginal pmf** of Y_1 is

$$p_{Y_1}(y_1) = \sum_{y_2} p_{Y_1,Y_2}(y_1, y_2)$$

and the **marginal pmf** of Y_2 is

$$p_{Y_2}(y_2) = \sum_{y_1} p_{Y_1,Y_2}(y_1, y_2).$$

MAIN POINT: In the two-dimensional discrete case, marginal pmfs are obtained by "summing over" the other variable.

TERMINOLOGY: Let (Y_1, Y_2) be a **continuous** random vector with pdf $f_{Y_1,Y_2}(y_1, y_2)$. Then the **marginal pdf** of Y_1 is

$$f_{Y_1}(y_1) = \int_{-\infty}^{\infty} f_{Y_1,Y_2}(y_1, y_2)dy_2$$

and the **marginal pdf** of Y_2 is

$$f_{Y_2}(y_2) = \int_{-\infty}^{\infty} f_{Y_1,Y_2}(y_1, y_2)dy_1.$$

MAIN POINT: In the two-dimensional continuous case, marginal pdfs are obtained by "integrating over" the other variable.

Example 5.3. In a simple genetics model, the proportion, say Y_1, of a population with trait 1 is always less than the proportion, say Y_2, of a population with trait 2. Suppose that the random vector (Y_1, Y_2) has joint pdf

$$f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} 6y_1, & 0 < y_1 < y_2 < 1 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the marginal distributions $f_{Y_1}(y_1)$ and $f_{Y_2}(y_2)$.

SOLUTION. To find $f_{Y_1}(y_1)$, we integrate $f_{Y_1,Y_2}(y_1, y_2)$ over y_2. For $0 < y_1 < 1$,

$$f_{Y_1}(y_1) = \int_{y_2=y_1}^{1} 6y_1 dy_2 = 6y_1(1 - y_1).$$

Thus, the marginal distribution of Y_1 is given by

$$f_{Y_1}(y_1) = \begin{cases} 6y_1(1 - y_1), & 0 < y_1 < 1 \\ 0, & \text{otherwise.} \end{cases}$$

That is, $Y_1 \sim \text{beta}(2, 2)$. To find $f_{Y_2}(y_2)$, we integrate $f_{Y_1,Y_2}(y_1, y_2)$ over y_1. For values of $0 < y_2 < 1$,

$$f_{Y_2}(y_2) = \int_{y_1=0}^{y_2} 6y_1 dy_1 = 3y_1^2\Big|_0^{y_2} = 3y_2^2.$$

Thus, the marginal distribution of Y_2 is given by

$$f_{Y_2}(y_2) = \begin{cases} 3y_2^2, & 0 < y_2 < 1 \\ 0, & \text{otherwise.} \end{cases}$$

That is, $Y_2 \sim \text{beta}(3, 1)$.

$= P(Y_2 > 1/2)$

(b) Find the probability that the proportion of individuals with trait 2 exceeds 1/2.

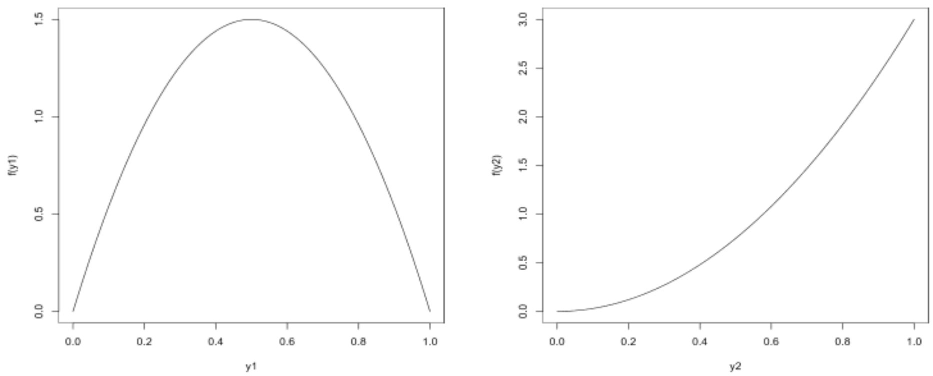
SOLUTION. Here, we want to find $P(B)$, where the set

$$B = \{(y_1, y_2) : 0 < y_1 < y_2,\ y_2 > 1/2\}. \quad = \left\{ (y_1, y_2) : \begin{array}{l} 0 < y_1 < y_2 < 1 \\ \frac{1}{2} < y_2 < 1 \end{array} \right\}$$

This probability can be computed two different ways:

(i) using the **joint** distribution $f_{Y_1,Y_2}(y_1, y_2)$ and computing

$$P[(Y_1, Y_2) \in B] = \int_{y_2=1/2}^{1} \int_{y_1=0}^{y_2} 6y_1 \, dy_1 dy_2.$$



$Y_1 \sim \text{beta}(2,2)$ $\qquad$ $Y_2 \sim \text{beta}(3,1)$

Figure 5.16: *Marginal distributions in Example* 5.3.

(ii) using the **marginal** distribution $f_{Y_2}(y_2)$ and computing

$$P(Y_2 > 1/2) = \int_{y_2=1/2}^{1} 3y_2^2 dy_2.$$

Either way, you will get the same answer! Notice that in (i), you are computing the volume under $f_{Y_1,Y_2}(y_1,y_2)$ over the set B. In (ii), you are finding the area under $f_{Y_2}(y_2)$ over the set $\{y_2 : y_2 > 1/2\}$.

(c) Find the probability that the proportion of individuals with trait 2 is at least twice that of the proportion of individuals with trait 1.

SOLUTION. Here, we want to compute $P(Y_2 \geq 2Y_1)$; i.e., we want to compute $P(D)$, where the set

$$D = \{(y_1, y_2) : y_2 \geq 2y_1\}.$$

$$B = \left\{ (y_1, y_2) : \begin{array}{l} y_2 \geq 2y_1 \\ 0 < y_1 < y_2 < 1 \end{array} \right\} = \left\{ (y_1, y_2) : \begin{array}{l} 0 < y_1 < \frac{y_2}{2} \\ 0 < y_2 < 1 \end{array} \right\}$$

This equals

$$P[(Y_1, Y_2) \in D] = \int_{y_2=0}^{1} \int_{y_1=0}^{y_2/2} 6y_1 dy_1 dy_2 = 0.25.$$

This is the volume under $f_{Y_1,Y_2}(y_1,y_2)$ over the set D. $\square$