Section 5.7 Expectations of functions of random variables

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Section 5.7
Expectati...

We want to compute $P(B)$. Note that

$$\overline{B} = \{\text{all 20 responses are less than } \mu + 2\sigma\}$$

and recall that $P(B) = 1 - P(\overline{B})$. We will compute $P(\overline{B})$ because it is easier. The probability that the first patient's response Y_1 is less than $\mu + 2\sigma$ is given by

$$F_{Y_1}(\mu + 2\sigma) = P(Y_1 < \mu + 2\sigma) = P(Z < 2) = F_Z(2) = 0.9772,$$

where $Z \sim \mathcal{N}(0,1)$ and $F_Z(\cdot)$ denotes the standard normal cdf. This probability is same for each patient, because each patient's response follows the same $\mathcal{N}(\mu, \sigma^2)$ distribution. Because the patients' responses are independent random variables,

$$\begin{aligned} P(\overline{B}) &= P(Y_1 < \mu + 2\sigma, Y_2 < \mu + 2\sigma, ..., Y_{20} < \mu + 2\sigma) \\ &= \prod_{i=1}^{20} F_{Y_i}(\mu + 2\sigma) \\ &= [F_Z(2)]^{20} \approx 0.63. \end{aligned}$$

Finally, $P(B) = 1 - P(\overline{B}) \approx 1 - 0.63 = 0.37$. $\square$

5.7 Expectations of functions of random variables

RESULT: Suppose that $\boldsymbol{Y} = (Y_1, Y_2, ..., Y_n)$ has joint pdf $f_{\boldsymbol{Y}}(\boldsymbol{y})$, or joint pmf $p_{\boldsymbol{Y}}(\boldsymbol{y})$, and suppose that $g(\boldsymbol{Y}) = g(Y_1, Y_2, ..., Y_n)$ is a real vector valued function of $Y_1, Y_2, ..., Y_n$; i.e., $g : \mathcal{R}^n \rightarrow \mathcal{R}$. Then,

- if $\boldsymbol{Y}$ is discrete,

$$E[g(\boldsymbol{Y})] = \sum_{y_1} \sum_{y_2} \cdots \sum_{y_n} g(\boldsymbol{y}) p_{\boldsymbol{Y}}(\boldsymbol{y}),$$

- and if $\boldsymbol{Y}$ is continuous,

$$E[g(\boldsymbol{Y})] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g(\boldsymbol{y}) f_{\boldsymbol{Y}}(\boldsymbol{y}) d\boldsymbol{y}.$$

If these quantities are not finite, then we say that $E[g(\boldsymbol{Y})]$ does not exist.

PROPERTIES OF EXPECTATIONS: Let $\boldsymbol{Y} = (Y_1, Y_2, ..., Y_n)$ be a discrete or continuous random vector, suppose that $g, g_1, g_2, ..., g_k$ are real vector valued functions from $\mathcal{R}^n \to \mathcal{R}$, and let c be any real constant. Then,

(a) $E(c) = c$

(b) $E[cg(\boldsymbol{Y})] = cE[g(\boldsymbol{Y})]$

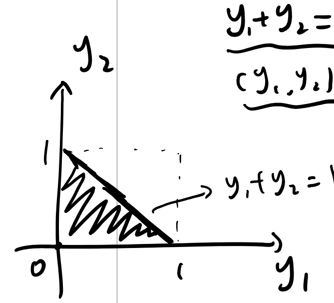
(c) $E[\sum_{j=1}^k g_j(\boldsymbol{Y})] = \sum_{j=1}^k E[g_j(\boldsymbol{Y})]$.

Example 5.12. In Example 5.6, Y_1 denotes the amount of grain 1 in stock and Y_2 denotes the amount of grain 2 in stock. Both Y_1 and Y_2 are measured in 100s of lbs. The joint distribution of Y_1 and Y_2 is

$y_1 + y_2 = 1$

(y_1, y_2)

$$f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} 24y_1y_2, & y_1 > 0,\ y_2 > 0,\ 0 < y_1 + y_2 < 1 \\ 0, & \text{otherwise.} \end{cases}$$



What is the expected total amount of grain $(Y_1 + Y_2)$ in stock?

SOLUTION. Let the function $g : \mathcal{R}^2 \to \mathcal{R}$ be defined by $g(y_1, y_2) = y_1 + y_2$. We would like to compute $E[g(Y_1, Y_2)] = E(Y_1 + Y_2)$. From the last result, we know that

$$\begin{aligned} E(Y_1 + Y_2) &= \int_{y_1=0}^1 \int_{y_2=0}^{1-y_1} (y_1 + y_2) \times 24y_1y_2\, dy_2dy_1 \\ &= \int_{y_1=0}^1 \int_{y_2=0}^{1-y_1} (24y_1^2y_2 + 24y_1y_2^2)\, dy_2dy_1 \\ &= \int_{y_1=0}^1 \left[\left(24y_1^2 \frac{y_2^2}{2} \Big|_0^{1-y_1} \right) + \left(24y_1 \frac{y_2^3}{3} \Big|_0^{1-y_1} \right) \right] dy_1 \\ &= \int_{y_1=0}^1 12y_1^2(1-y_1)^2 dy_1 + \int_{y_1=0}^1 8y_1(1-y_1)^3 dy_1 \\ &= 12 \int_{y_1=0}^1 y_1^2(1-y_1)^2 dy_1 + 8 \int_{y_1=0}^1 y_1(1-y_1)^3 dy_1 \\ &= 12 \left[\frac{\Gamma(3)\Gamma(3)}{\Gamma(6)} \right] + 8 \left[\frac{\Gamma(2)\Gamma(4)}{\Gamma(6)} \right] = 4/5. \end{aligned}$$

Vector Calculus.

$E(Y_1 \cdot Y_2)$

$E(\sin(Y_1 + Y_2))$

The expected total amount of grain in stock is 80 lbs.

REMARK: In the calculation above, we twice used the fact that

$$\int_0^1 y^{\alpha-1}(1-y)^{\beta-1} dy = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}.$$

$E(Y_1+Y_2)$
$= E(Y_1) + E(Y_2)$

ANOTHER SOLUTION: To compute $E(Y_1+Y_2)$, we could have taken a different route. In Example 5.6, we discovered that the marginal distributions were

$$\begin{aligned} Y_1 &\sim \text{beta}(2,3) \\ Y_2 &\sim \text{beta}(2,3) \end{aligned}$$

so that

$$E(Y_1) = E(Y_2) = \frac{2}{2+3} = \frac{2}{5}.$$

Because expectations are linear, we have

$$E(Y_1+Y_2) = \frac{2}{5} + \frac{2}{5} = \frac{4}{5}. \ \square$$

RESULT: Suppose that Y_1 and Y_2 are **independent** random variables. Let $g(Y_1)$ be a function of Y_1 only, and let $h(Y_2)$ be a function of Y_2 only. Then,

$$E[g(Y_1)h(Y_2)] = E[g(Y_1)]E[h(Y_2)],$$

provided that all expectations exist.

Proof. Without loss, assume that (Y_1, Y_2) is a continuous random vector (the discrete case is analogous). Suppose that (Y_1, Y_2) has joint pdf $f_{Y_1,Y_2}(y_1, y_2)$ with support $R \subset \mathcal{R}^2$. Note that

If Y_1 and Y_2 are indepe

$E[\sin(Y_1) \cos(Y_2)]$
$= E[\sin(Y_1)] \cdot E[\cos(Y_2)]$

$$\begin{aligned} E[g(Y_1)h(Y_2)] &= \int_{\mathcal{R}^2} g(y_1)h(y_2)f_{Y_1,Y_2}(y_1,y_2)dy_2dy_1 \\ &= \int_{\mathcal{R}}\int_{\mathcal{R}} g(y_1)h(y_2)f_{Y_1}(y_1)f_{Y_2}(y_2)dy_2dy_1 \\ &= \int_{\mathcal{R}} g(y_1)f_{Y_1}(y_1)dy_1 \int_{\mathcal{R}} h(y_2)f_{Y_2}(y_2)dy_2 \\ &= E[g(Y_1)]E[h(Y_2)]. \ \square \end{aligned}$$

COROLLARY: If Y_1 and Y_2 are **independent** random variables, then

$$E(Y_1Y_2) = E(Y_1)E(Y_2).$$

This is a special case of the previous result obtained by taking $g(Y_1) = Y_1$ and $h(Y_2) = Y_2$.