CHAPTER 5 PROBLEMS

Note: See the Exam 1/Exam 2 Practice Problems handouts for problems from Chapters 2-4 (WMS).

1. When a current Y_1 (measured in amperes) flows through a resistance Y_2 (measured in ohms), the power generated is given $W = Y_1^2 Y_2$ (measured in watts). Suppose that the marginal distributions of Y_1 and Y_2, respectively, are given by

$$f_{Y_1}(y_1) = \begin{cases} 6y_1(1-y_1), & 0 < y_1 < 1, \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad f_{Y_2}(y_2) = \begin{cases} 2y_2, & 0 < y_2 < 1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) The random variables Y_1 and Y_2 both have beta distributions. Give the parameters associated with each distribution.
(b) Using the above model for Y_2, compute the probability that the resistance is less than 0.5 ohms.
(c) Find $E(Y_1^3)$.
(d) Assuming that Y_1 and Y_2 are independent, compute $E(W)$.

2. The random vector (Y_1, Y_2) has joint pdf

$$f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} \frac{6}{7}\left(y_1^2 + \frac{y_1 y_2}{2}\right), & 0 < y_1 < 1,\ 0 < y_2 < 2 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find $f_{Y_2|Y_1}(y_2|y_1)$, the conditional distribution of Y_2 given $Y_1 = y_1$.
(b) Compute $P(Y_1 > Y_2)$.
(c) Compute $\text{Cov}(Y_1, Y_2)$.

3. An engineering system consists of two components operating independently of each other. Let Y_1 denote the time until component 1 fails, and let Y_2 denote the time until component 2 fails. An engineer models Y_1 as an exponential random variable with mean 1, and Y_2 as a gamma random variable with $\alpha = \beta = 2$.
(a) Write out the joint distribution of $\boldsymbol{Y} = (Y_1, Y_2)$. Make sure to note your support.
(b) Find the probability that the lifetime of component 1 exceeds the lifetime of component 2. That is, find $P(Y_1 > Y_2)$.
(c) Find $P(Y_2 > Y_1 | Y_2 \leq 2Y_1)$.

4. Suppose that Y_1, Y_2, and Y_3 are random variables with

$$\begin{array}{ccc} E(Y_1) = 1 & E(Y_2) = 2 & E(Y_3) = 3 \\ V(Y_1) = 1 & V(Y_2) = 4 & V(Y_3) = 9 \\ \text{Cov}(Y_1, Y_2) = 0 & \text{Cov}(Y_1, Y_3) = 1 & \text{Cov}(Y_2, Y_3) = -1. \end{array}$$

Define the linear combination $U = Y_1 - 2Y_2 + Y_3$. Find the mean and variance of U.

5. A large insurance agency services a number of customers who have purchased both a homeowner's policy and an automobile policy from the agency. For each type of policy a deductible must be specified. For the homeowner policy, the choices are \$100, \$250, and \$500. For the automobile policy, the choices are \$0, \$100, \$250, and \$500. For a group of customers, let X_1 and X_2 denote the homeowner policy deductible and automobile policy deductible, respectively. Actuaries have provided us with the joint distribution of $\boldsymbol{X} = (X_1, X_2)$, depicted in the following table.

	$x_2 = 0$	$x_2 = 100$	$x_2 = 250$	$x_2 = 500$
$x_1 = 100$	0.02	0.10	0.10	0.08
$x_1 = 250$	0.12	0.12	0.10	0.06
$x_1 = 500$	0.06	0.08	0.10	0.06

(a) Find the marginal distribution of X_1.
(b) Find the conditional distribution of X_2, given $x_1 = 250$.
(c) Find the conditional mean and variance of X_1, given $x_2 = 0$.

6. Suppose that the random vector (Y_1, Y_2) has joint pdf

$$f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} c, & 0 < y_1 < 2,\ 0 < y_2 < 1,\ 2y_2 < y_1 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Sketch the support region of (Y_1, Y_2) in the (y_1, y_2) plane. Place y_1 on the horizontal axis and y_2 on the vertical axis.
(b) Find the value of c that makes $f_{Y_1,Y_2}(y_1, y_2)$ a valid pdf.
(c) Describe, in words, what the function $f_{Y_1,Y_2}(y_1, y_2)$ looks like.
(d) Compute $\text{Cov}(Y_1, Y_2)$.
(e) Find the conditional distribution of Y_2, given Y_1. Find the mean and variance of this conditional distribution.

7. An electronic device is designed to switch house lights on and off at random times after it has been activated. Assume that the device is designed in such a way that it will be switched on and off exactly once in a one-hour period. Let X denote the time (in **hours**) at which the lights are turned on, and let Y denote the time (in hours) at which the lights are turned off. The joint pdf for (X, Y) is given by

$$f_{X,Y}(x, y) = \begin{cases} 8xy, & 0 < x < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Are X and Y are independent? Prove your answer.
(b) What is the probability that the device will turn on and turn off in less than 30 minutes after it has been activated?

(c) Find $P(Y - X < 0.2)$.
(d) Find the correlation between X and Y.

8. Suppose that X_1, X_2, and X_3 are random variables. Prove that

$$\text{Cov}(X_1, X_2 + X_3) = \text{Cov}(X_1, X_2) + \text{Cov}(X_1, X_3).$$

9. In a simple genetics model, the proportion, say X, of a population with Trait 1 is always less than the proportion, say Y, of a population with trait 2. Suppose that (X, Y) has joint pdf

$$f_{X,Y}(x, y) = \begin{cases} 6x, & 0 < x < y < 1 \\ 0, & \text{otherwise.} \end{cases}$$

(a) If subjects in the population possessing trait 1 always possess trait 2, then $Y - X$ denotes the proportion of the population which has trait 2, but not trait 1. Compute $E(Y - X)$.
(b) Find the conditional distribution of Y, given X.
(c) Find the conditional mean and variance of Y, given X.

10. Suppose X_1 and X_2 are independent random variables with $E(X_1) = E(X_2) = 0$, $V(X_1) = 1$, and $V(X_2) = 4$. Define

$$\begin{aligned} U_1 &= X_1 + X_2 \\ U_2 &= X_1 - X_2. \end{aligned}$$

Compute ρ_{U_1,U_2}, the correlation between U_1 and U_2.

11. The management at a fast-food outlet is interested in the joint behavior of the random variables Y_1 and Y_2. The variable Y_1 denotes the total time between a customers arrival at the store and his/her departure from the service window. The variable Y_2 denotes the time a customer waits in line before reaching the service window. Because Y_1 includes the time a customer waits in line, we must have $Y_2 \leq Y_1$. Both Y_1 and Y_2 are measured in minutes. The joint distribution of (Y_1, Y_2) is given by

$$f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} e^{-y_1}, & 0 < y_2 < y_1 < \infty \\ 0, & \text{otherwise.} \end{cases}$$

(a) Sketch the support of the random vector (Y_1, Y_2) in the y_1-y_2 plane. Put y_1 on the horizontal axis.
(b) The quantity $Y_1 - Y_2$ denotes the time (in minutes) spent at the service window. Compute $P(Y_1 - Y_2 > 1)$.
(c) Find $E(Y_1 - Y_2)$.
(d) If 2 minutes elapse between a customer's arrival at the store and his departure from the service window, find the probability that he waited in line less than 1 minute to reach the window. That is, compute $P(Y_2 < 1 | Y_1 = 2)$.