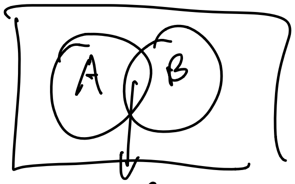
1. $P(A) = 0.2 \qquad P(B) = 0.3 \qquad P(A \cap B) = 0.1$



$A \cap B$

(a) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= 0.2 + 0.3 - 0.1 = \underline{0.4}$

(b) $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)} = 1/3$

(c) $P(A \cap \bar{B}) = P(A) - P(A \cap B) = 0.2 - .1 = .1$

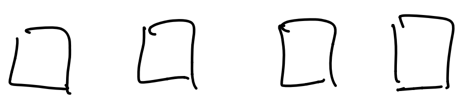
(d) $P(B \mid A \cup B) = \dfrac{P(B \cap (A \cup B))}{P(A \cup B)}$

$= \dfrac{P(B)}{P(A \cup B)} = \dfrac{0.3}{0.4} = .75$

2. (a) $Y \sim \text{Poisson}(\lambda = 2.7)$

$P(Y=0) = \text{poissonpdf}(y=0, \lambda=2.7) = e^{-\lambda} \cdot \dfrac{\lambda^0}{0!}$

$= e^{-2.7} = 0.0672$

(b) 

"success" as "no bacteria colonies"

Denote X as the # of specimens that contain no bacteria colonies out of four

$X \sim$ Binomial(($n=4$, $p = P$(no bacteria colonies in one specimen)))

$p = P(Y=0) = 0.0672$

$X \sim \text{Binomial}(n=4, p=0.0672)$

$P(X \geq 1) = 1 - P(X=0) = 1 - \text{binompdf}(4, 0.0672, 0)$
$= 0.2429.$

(c). W: # to first "success" → no bacteria colonies

$W \sim \text{Geometric}(P = 0.0672)$

pmf: $P_W(w) = (1-p)^{w-1} \times p = (1-0.0672)^{w-1} \times 0.0672$
for $w = 1, 2, 3, \ldots$

(d) $P(W > 10) = 1 - P(W \leq 10)$
$= 1 - \text{geomcdf}(0.0672, 10)$
$= 0.4988$

$P(W > 10) = (1-p)^{10} = (1-0.0672)^{10} = 0.4988$

3. (a) $F_Y(y) = P(Y \leq y) = 0$ if $y < 0$

if $y > 0$ $= \int_0^y f_Y(v)\,dv = \int_0^y v^2 e^{-v^3/3}\,dv$

$u = v^3/3$ $\quad du = v^2 dv$

3.

$$= \int_0^{y^3/3} e^{-u}\,du = (-e^{-u})\Big|_0^{y^3/3} = 1 - e^{-y^3/3}$$

$$F_Y(y) = \begin{cases} 1 - e^{-y^3/3} & y > 0 \\ 0 & y \leq 0 \end{cases}$$

(b) $P(Y<1) = F_Y(1) = 1 - e^{-1/3}$

(c) $E(Y) = \int y f_Y(y)\,dy$

$$= \int_0^\infty y \; y^2 e^{-y^3/3}\,dy$$

$$= \int_0^\infty y^3 e^{-y^3/3}\,dy$$

$u = y^3/3 \qquad du = \underline{y^2 dy} \qquad y = (3u)^{1/3}$

$$= \int_0^\infty e^{-u} \, y \times y^2 dy$$

$$= \int_0^\infty e^{-u} (3u)^{1/3}\,du$$

$$= 3^{1/3} \int_0^\infty u^{1/3} e^{-u}\,du$$

$$= 3^{1/3} \times \Gamma(1+\tfrac{1}{3})\, 1^{(1+\frac{1}{3})} = 3^{1/3}\,\Gamma(\tfrac{4}{3})$$

$$\Gamma(\alpha)\beta^\alpha = \int_0^\infty y^{\alpha-1} e^{-y/\beta}\,dy$$

$V(Y) = E[Y^2] - E[Y]^2 = 3^{2/3}\Gamma(\frac{5}{3}) - [3^{1/3}\Gamma(\frac{4}{3})]^2$

$$E[Y^2] = \int_0^{+\infty} y^2 \; y^2 e^{-y^3/3} dy$$

$$= \int_0^{+\infty} y^4 e^{-y^3/3} dy$$

$u = y^3/3$

$du = y^2 dy$

$y = (3u)^{1/3}$

$$= \int_0^{+\infty} y^2 \times e^{-u} \times y^2 dy$$

$$= \int_0^{+\infty} (3u)^{2/3} \times e^{-u} du$$

$$= 3^{2/3} \int_0^{+\infty} u^{2/3} e^{-u} du$$

$$= 3^{2/3} \Gamma(\frac{2}{3}+1) \, 1^{(\frac{2}{3}+1)} = 3^{2/3}\Gamma(\frac{5}{3})$$

6. (a) $Cov(Y_1 + Y_2, Y_1 - Y_3)$

$= Cov(Y_1, Y_1) + Cov(Y_2, Y_1) - Cov(Y_1, Y_3) - Cov(Y_2, Y_3)$

$= Var(Y_1) + Cov(Y_1, Y_2) - 0 - 1$

$= 1 - 1 - 0 - 1 = -1$

(b) $U = 2Y_1 - Y_2 + Y_3$

$E[U] = E[2Y_1 - Y_2 + Y_3]$

$= 2E[Y_1] - E[Y_2] + E[Y_3]$

$= 2 \times 0 - 1 - 1 = -2$

$Var(U) = Cov(U, U)$

$= Cov(2Y_1 - Y_2 + Y_3, \; 2Y_1 - Y_2 + Y_3)$

$$= Cov(2Y_1, 2Y_1) - Cov(2Y_1, Y_2) + Cov(2Y_1, Y_3)$$
$$- Cov(Y_2, 2Y_1) + Cov(Y_2, Y_2) - Cov(Y_2, Y_3)$$
$$+ Cov(Y_3, 2Y_1) - Cov(Y_3, Y_2) + Cov(Y_3, Y_3)$$
$$= 2^2 Var(Y_1) - 2Cov(Y_1, Y_2) + 2Cov(Y_1, Y_3)$$
$$- 2Cov(Y_1, Y_2) + Var(Y_2) - Cov(Y_2, Y_3)$$
$$+ 2Cov(Y_1, Y_3) - Cov(Y_2, Y_3) + Var(Y_3)$$
$$= \cdots\cdots$$

7. (a) $\underline{m_{Y_1,Y_2}(t_1,t_2)} = E[e^{t_1Y_1+t_2Y_2}]$

$$E[Y_1Y_2] = \left.\left(\frac{\partial^2 m_{Y_1Y_2}(t_1,t_2)}{\partial t_1 \partial t_2}\right)\right|_{t_1=t_2=0}$$

$$\frac{\partial m_{Y_1Y_2}(t_1,t_2)}{\partial t_1} = \frac{\partial}{\partial t_1} E[e^{t_1Y_1+t_2Y_2}]$$
$$= E\left[\frac{\partial}{\partial t_1} e^{(t_1Y_1+t_2Y_2)}\right]$$
$$= \underline{E[Y_1 e^{(t_1Y_1+t_2Y_2)}]}$$

$$\frac{\partial^2 m_{Y_1Y_2}(t_1,t_2)}{\partial t_1 \partial t_2} = \frac{\partial}{\partial t_2}\left(\frac{\partial m_{Y_1Y_2}(t_1,t_2)}{\partial t_1}\right)$$
$$= \frac{\partial}{\partial t_2} E(Y_1 e^{t_1Y_1+t_2Y_2})$$
$$= E\left(\frac{\partial}{\partial t_2} Y_1 e^{t_1Y_1+t_2Y_2}\right)$$
$$= E[Y_1Y_2 \underline{e^{t_1Y_1+t_2Y_2}}]$$

set $t_1 = t_2 = 0$, $\left.\dfrac{\partial^2 m_{Y_1,Y_2}(t_1,t_2)}{\partial t_1 \partial t_2}\right|_{t_1=0,\,t_2=0}$

$$= E[Y_1 Y_2 \times e^{0+0}] = E[Y_1 Y_2]$$

(b) If Y_1, Y_2 are independent

then $E[g(Y_1) \times h(Y_2)] = E[g(Y_1)] \times E[h(Y_2)]$.

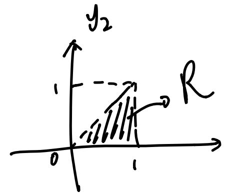
So $m_{Y_1,Y_2}(t_1,t_2) = E[e^{t_1 Y_1 + t_2 Y_2}]$

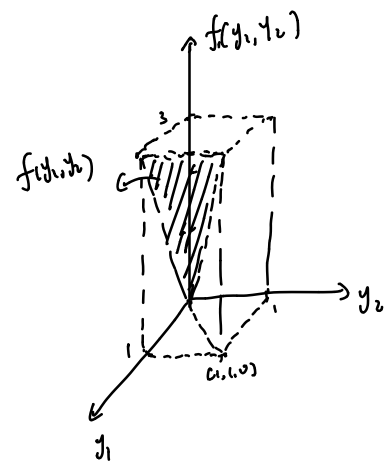
$$= E[e^{t_1 Y_1} \times e^{t_2 Y_2}]$$

$$= E[e^{t_1 Y_1}] \times E[e^{t_2 Y_2}]$$

$$= M_{Y_1}(t_1) \times M_{Y_2}(t_2)$$

4. a





b.

$$f_{Y_2}(y_2) = \int_{y_2}^{1} 3y_1 \, dy_1 = \frac{3}{2} y_1^2 \Big|_{y_2}^{1} = \frac{3}{2}(1 - y_2^2), \quad 0 < y_2 < 1$$

$$f_{Y_1|Y_2}(y_1|y_2) = \frac{f_{Y_1,Y_2}(y_1,y_2)}{f_{Y_2}(y_2)} = \frac{3y_1}{\frac{3}{2}(1-y_2^2)} = \frac{2y_1}{1-y_2^2} \qquad y_2 < y_1 < 1$$

$$f_{Y_1|Y_2=\frac{1}{2}}(y_1|\tfrac{1}{2}) = \frac{2y_1}{1-(\frac{1}{2})^2} = \frac{8y_1}{3}, \quad \frac{1}{2}<y_1<1$$

c. $$f_{Y_1}(y_1) = \int_0^{y_1} 3y_1\,dy_2 = 3y_1\left(y_2\Big|_0^{y_1}\right) = 3y_1^2, \quad 0<y_1<1$$

$$P(Y_1 > \tfrac{3}{4}) = \int_{\frac{3}{4}}^1 f_{Y_1}(y_1)\,dy_1$$
$$= \int_{3/4}^1 3y_1^2\,dy_1 = y_1^3\Big|_{3/4}^1 = 1-(\tfrac{3}{4})^3$$

$$P(Y_1 > 3/4 \mid Y_2=\tfrac{1}{2}) = \int_{3/4}^1 f_{Y_1|Y_2=1/2}(y_1|\tfrac{1}{2})\,dy_1$$
$$= \int_{3/4}^1 \frac{8y_1}{3}\,dy_1$$
$$= \frac{4}{3}y_1^2\Big|_{3/4}^1 = \frac{4}{3}\times\left(1-(\tfrac{3}{4})^2\right)$$

(d) No, the support of $f_{Y_1Y_2}(y_1,y_2)$ is $\{(y_1,y_2): 0<y_2<y_1<1\}$
y_2 increases, so does y_1

or. $f_{Y_1Y_2}(y_1,y_2) = 3y_1$, $0<y_2<y_1<1$
$f_{Y_1}(y_1)\times f_{Y_2}(y_2) = 3y_1^2 \times \frac{3}{2}(1-y_2^2) = \frac{9}{2}y_1^2(1-y_2^2)$, $0<y_1<1$, $0<y_2<1$
are not equivalent.

5. (a) Let $g(y_1) = \begin{cases} 1/8\, y_1 e^{-y_1/2} & y_1>0 \\ 0 & \text{otherwise} \end{cases}$

$h(y_2) = \begin{cases} e^{-y_2/2}, & y_2>0 \\ 0 & \text{otherwise} \end{cases}$

$$f_{Y_1,Y_2}(y_1,y_2) = g(y_1)\,h(y_2)$$

So Y_1, Y_2 are independent

(b) $$f_{Y_1}(y_1) = \int_0^{+\infty} \frac{1}{8} y_1 e^{-\frac{y_1+y_2}{2}} dy_2$$

$$= \frac{1}{8} y_1 e^{-y_1/2} \int_0^{+\infty} e^{-y_2/2} dy_2$$

$$= \frac{1}{4} y_1 e^{-y_1/2}, \quad y_1 > 0$$

$$f_{Y_2}(y_2) = \int_0^{+\infty} \frac{1}{8} y_1 e^{-\frac{y_1+y_2}{2}} dy_1$$

$$= \frac{1}{8} e^{-\frac{y_2}{2}} \int_0^{+\infty} y_1 e^{-\frac{y_1}{2}} dy_1 \qquad y_2 > 0$$

using $\Gamma(\alpha)\beta^{\alpha} = \int_0^{+\infty} y^{\alpha-1} e^{-y/\beta} dy$

yields $\int_0^{+\infty} y_1 e^{-y_1/2} dy_1 = \Gamma(2)\, 2^2 = 4$

So $f_{Y_2}(y_2) = \frac{1}{2} e^{-y_2/2} \qquad y_2 > 0$

$\Rightarrow Y_1 \sim \text{Gamma}(2,2), \quad Y_2 \sim \text{Exp}(2)$

$$E(Y_1+Y_2) = E(Y_1)+E(Y_2) = 2\times 2+2 = 6$$

$$V(Y_1+Y_2) = V(Y_1)+V(Y_2) + \underbrace{2\,\mathrm{cov}(Y_1,Y_2)}_{=0}$$

$\to$ since Y_1, Y_2 are independent

$$= V(Y_1) + V(Y_2)$$

$$= 2\times 2^2 + 4 = 12$$

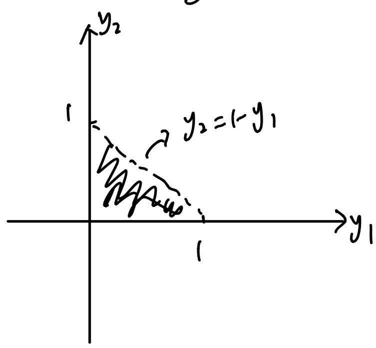
(c) $E\left(\frac{Y_2}{Y_1}\right) = \iint \frac{y_2}{y_1} f_{Y_1 Y_2}(y_1, y_2)\, dy_1 dy_2$

$$= \int_0^{+\infty}\int_0^{+\infty} \frac{y_2}{y_1} \frac{1}{8} y_1 e^{-\frac{y_1+y_2}{2}} dy_1 dy_2$$

$$= \frac{1}{8}\int_0^{+\infty} e^{-\frac{y_1}{2}} dy_1 \times \int_0^{+\infty} y_2 e^{-\frac{y_2}{2}} dy_2$$

$$= \frac{1}{8} \times 2 \times 4 = 1$$

(d)



$$\int_0^1 \int_0^{1-y_1} \frac{1}{8} y_1 e^{-\frac{y_1+y_2}{2}} dy_2 dy_1$$